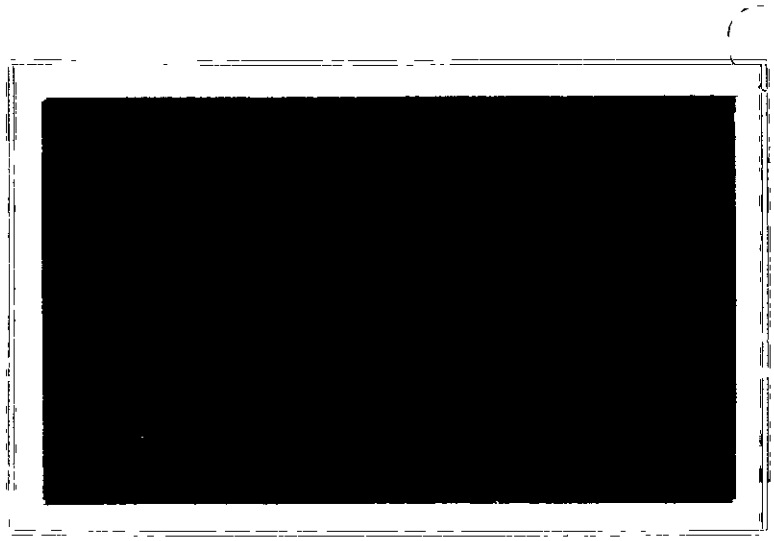
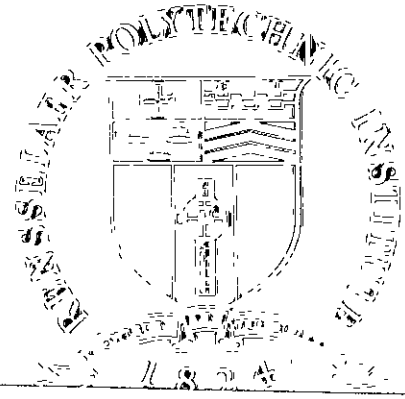
2 (mix)





Rensselaer Polytechnic Institute

Troy, New York

Rensselaer Polytechnic Institute

Troy, New York 12181

Final Report - Vol. I

Contract No. NAS8-21131

Covering Period Nov. 4, 1969 - April 4, 1970

NATIONAL AERONAUTICS AND
SPACE ADMINISTRATION

Optimal Time Invariant Output Feedback Controllers

by Derek E. McBrinn

Submitted on behalf of

Rob Roy
Professor
Systems Engineering Division

CONTENTS

LIST OF FIGURES

LIST OF SYMBOLS

Symbol	Meaning	Page First Used
T	Terminal time for optimization problem	2
V	Covariance matrix	39
W	Matrix of necessary conditions	44
t	Time	2
u	Scalar control	56
$\underline{u}(t)$	Control vector	2
$\underline{v}^T$, $\underline{v}_i^T$	Row eigenvector of a matrix	14
$\underline{w}$, $\underline{w}_i$	Column eigenvector of a matrix	14
$\underline{x}(t)$	State vector	2
$\hat{\underline{x}}(t)$	Subset of state vector	9
$\underline{y}(t)$	Output vector	8
$\underline{y}(t)$	Alternative state vector	40
$\underline{z}(t)$	Alternative state vector	9
α	Parameter of a matrix	14
Δ	Small perturbation	16
δ	Small perturbation	47
ϵ	Element of the set	2
$\phi(t)$	State transition matrix	36
Λ	Diagonal matrix of eigenvalues	51
λ, λ_i	Eigenvalue of a matrix	14
μ	Step size (convergence) parameter	16
∇	Gradient matrix	16
(t)	Function of time	2
$\cdot$ as in $\dot{\underline{x}}(t)$	Differentiation with respect to time	2
^{T} as in B^T	Transform of matrix or vector	2
$^{-1}$ as in R^{-1}	Inverse of matrix	2

Symbol	Meaning	Page First Used
$\square\square$ as in $\square K \square$	Vector arrangement of a matrix by column ordering	15
$[0, T]$	Closed time interval from 0 to T	2
ij as in k_{ij} or $[\ ]_{ij}$	Element in row i, column j of the matrix	10
i as in x_i	i^{th} component of vector	15
q as in $\underline{b}_q$	q^{th} column of the matrix B	14
$\Big\vert_T$	Evaluation at given point	33
$\Big\Vert\ \Big\Vert$	Euclidian norm	16
$\mathrm{Re}[\ ]$	Real part	15
$\mathrm{E}[\ ]$	Expected value	38
$\exp[\ ]$	Exponentiation	37
$\mathrm{tr}[\ ]$	Trace of matrix	33

ACKNOWLEDGMENT

The author gratefully acknowledges the advice and guidance given him by his thesis committee in the preparation of this dissertation. A particular debt is due to his advisor, Dr. Rob Roy, strategist extraordinaire, for a special blend of encouragement, exhortation and counsel. Thanks also go to the author's colleagues at R.P.I. for their willingness to enter technical discussions at any time; Mr. Lutz Willner must be singled out for special mention in this respect.

The manuscript was typed by Miss Kathy Reid, and her skill and patience are much appreciated.

This research was performed at Rensselaer Polytechnic Institute under the sponsorship of NASA Contract No. NAS8-21131. Financial assistance as also received from an N.D.E.A. Title IV Fellowship and from a grant by Ling Temco Vought, Inc.

ABSTRACT

The problem of the determination of optimal time-invariant output feedback controllers for linear dynamic systems with quadratic cost functionals is considered. Two distinct cases arise, depending on whether optimization is over a finite or a semi-infinite time interval.

For the semi-infinite time interval problem a gain initialization technique is derived to complement existing optimization techniques. The gain initialization technique determines the feedback gains required to (locally) maximize the system stability. A computational algorithm for the technique is incorporated in a digital computer program, and is used to stabilize a seven state model of a Saturn V booster rocket.

For the finite time interval problem a technique is derived to (locally) minimize the expected value of the cost functional. The technique uses the concept of Initial State Averaging. A computational algorithm is provided and incorporated in a digital computer program. The technique is illustrated by three examples.

CHAPTER I

INTRODUCTION

I.1 Motivation

During the past decade much excellent work has been done in the field of Optimal Control Theory. Many books and a great number of technical papers have been published - the Bibliography of this report cites only a small portion of the total. Not all of this work has been purely theoretical. A substantial background has been developed in the important practical area of computation and it has for several years been feasible to determine optimal controllers for a wide variety of physical systems.

It would be expected that practical applications of this theory would have appeared in abundance. Optimization is the essence of all good engineering design, and practicing engineers should seize upon any techniques which might aid them in their work. Application of the new techniques has, however, been disappointingly slow. With a few notable exceptions the practical design of control systems has remained based on the transfer-function techniques developed prior to the 1960's.

The explanation of the above paradox is widely recognized. It lies in the nature of the word "optimal", which is meaningless without a criterion of optimality. By and large, control theorists have used criteria of optimality dependent only on the performance of the control system. Design engineers, on the other hand, interpret "optimal" as embracing both system performance and system cost. The difference may be best illustrated by a simple example.

Consider the problem of designing a speed control device for a cheap movie camera. The objective is to ensure that sixteen frames of film are

exposed per second regardless of film tension, state of charge of the motor batteries, etc. The could be interpreted as comprising an example of the state-regulator problem in Optimal Control Theory, with the following method of analysis. The equations governing the film transportation are determined and linearized to the form

$$\dot{\underline{x}}(t) = A\,\underline{x}(t) + B\,\underline{u}(t) \qquad (I.1\text{-}1)$$

where $\underline{x}(t)$ is an NS-vector describing the deviation of the system from its desired state at time t

$\underline{u}(t)$ is an NC-vector defining the control inputs at time t

A and B are matrices whose elements depend on the characteristics of the system.

A cost functional is formulated having the form

$$J = \underline{x}^T(T)\,F\,\underline{x}(T) + \int_0^T \left[\underline{x}^T(\tau)\,Q\,\underline{x}(\tau) + \underline{u}^T(\tau)\,R\,\underline{u}(\tau)\right] d\tau \qquad (I.1\text{-}2)$$

where T is the duration of the scene to be filmed. The objective is to determine the control input $\underline{u}(t)$ for $t \in [0\ ,\ T]$ which minimizes the cost functional J. The positive semi-definite matrices F and Q are chosen to penalize any given control system for the deviations it allows from the desired state. The positive definite matrix R is a penalty for improvident usage of control power (i.e., battery energy). The solution to the problem defined by (I.1-1) and (I.1-2) can be shown[2] to be

$$\underline{u}(t) = -R^{-1}\,B^T\,D(t)\,\underline{x}(t) \qquad (I.1\text{-}3)$$

where $D(t)$ satisfies the matrix Riccati equation

$$\dot{D}(t) = -D(t)\,A - A^T D(t) + D(t)\,BR^{-1}\,B^T D(t) - Q \qquad (I.1\text{-}4)$$

with

$$D(T) = F \tag{I.1-5}$$

Defining

$$K^T(t) = R^{-1} B^T D(t) \tag{I.1-6}$$

and re-writing (I.1-3) as

$$\underline{u}(t) = -K^T(t)\, \underline{x}(t) \tag{I.1-7}$$

we see that the optimal control input can be generated as a time-varying linear combination of all of the states of the system. We note in passing that the less-fastidious photographer, willing to edit a few frames from the beginning and end of each scene and settle for good steady state performance, would be rewarded by simplification of the controller to the form[2]

$$\underline{u}(t) = -K^T\, \underline{x}(t) \tag{I.1-8}$$

This time-invariant controller would be much simpler to mechanize than that described by (I.1-7).

Now consider that speed control of the typical cheap movie camera is achieved by means of a simple flyball governor driving an on/off switch between the batteries and the motor. The flyball governor, invented by James Watt in 1788, was the first widely used automatic feedback controller.[5] In its simpler form it suffers from "hunting" about the set point. In its application to the movie camera it is driven by the single system state, motor speed. The single feedback gain can be considered as constant in the vicinity of the set point. It is clear that the flyball governor does not satisfy the requirements for an optimal controller as defined by (I.1-7).

Yet the flyball governor is indeed an optimal controller in the best engineering sense. It does an acceptable job at minimum cost. The mechanization of the "optimal" controller defined by (I.1-7) on the other hand, would be prohibitively expensive. It would require measurement and feedback of all system states. It would also require a digital computer to calculate and store the time-varying feedback gains for each scene to be filmed.

It should be clear from the above that the allowable degree of complexity of a control system is often constrained by considerations of cost. Weight, reliability and common engineering sense similarly often dictate simplicity. What the control system designer wants then is not that system which performs in the best possible manner. Rather he wishes the best possible performance for a given degree of complexity. This problem is considerably more difficult than that defined by (I.1-1) and (I.1-2). It must however, be solved if the benefits of optimal control are ever to be realized for the majority of potential applications.

This report considers a portion of the general problem of optimization within a given degree of control system complexity. Only linear systems with quadratic cost functionals are considered. Such combinations may be described by (I.1-1) and (I.1-2). The allowable degree of complexity of the controller is assumed to be time-invariant feedback of the system outputs only. The system outputs are those quantities which can be measured, and in most cases provide less than a full description of the system state at any time. The above limitations are those which are normally applied in classical control system analysis techniques. Thus this report seeks to optimize the classical controller, without increasing its complexity.

I.2 Historical Review

It is generally conceded that control system design prior to the Second World War was primarily an art. The techniques used were often empirical and thus confined to particular classes of problems. This situation was largely eliminated by the contributions of Nyquist,[6] Bode[7] and Evans[8] who established the theory and techniques of control system design which predominate in practical work even today. These techniques, however, are most suited to the design of relatively simple systems. They work best for systems having a single input and a single output related by a transfer function. Also these techniques are purely analytical and cannot be used to directly synthesize a control system.

The problem of synthesis was first tackled by Wiener[9], who considered the optimization of linear filters. This work was extended to control systems by Newton, Gould and Kaiser[10], who, however, still retained the transfer-function approach; their technique was to determine the values of system parameters required to minimize a cost functional for a specific perturbation of the system. It is significant that they made use of the Calculus of Variations in their approach to this problem. The extension to the classical Calculus of Variations provided by Pontryagin's Maximum Principle,[11] and the control problem framework provided by Kalman[12] finally brought Optimal Control Theory to maturity. This maturity, combined with the state-variable formulation presented by DeRusso, Roy and Close[1] and the increasing capability of digital computers, finally allowed the determination of optimal controllers for a wide variety of systems.

The early excitment at the disclosure of the ability to compute optimal controllers faded when it was realized that such controllers were very difficult to mechanize. The reasons for this difficulty are twofold. First, an optimal controller requires knowledge of the complete state of

the system. It may not, however, be feasible to measure all system states. Secondly, for finite time duration problems an optimal controller requires time-varying feedback.

There are two fundamental approaches to elimination of the first difficulty. The approach taken by Kalman and Bucy[13] was to estimate the unmeasurable states from the (noise corrupted) system outputs or measurable states. This approach was extended by Luenberger[14] and again by Ash[15]. Estimation of the unmeasurable states allows the mechanization of a regular optimal controller. Note, however, that it addes the complexities of a state estimator to those of the optimal controller.

The second approach to the problem of unmeasurable states is to design a controller which uses only the available information. Such a controller will be "sub-optimal" in the mathematical sense of minimizing a cost functional such as that described by (I.1-2). It may well, however, be optimal in the engineering sense. This approach has been tried by a number of people. Newton, Gould and Kaiser[10] essentially advocated such controllers, but optimized them for specific system disturbances. Consideration of only a specific disturbance is equivalent to imposing a specific initial condition on the system to be controlled. Max-min techniques were developed in an attempt to eliminate the dependence of the solution on the chosen initial condition.[16,17] In the max-min procedure the maximum system cost with respect to a set of initial conditions is minimized with respect to the system feedback gains. It still remains to choose an appropriate set of initial conditions. Rekasius[18] circumvented this problem by considering the initial condition giving the maximum ratio of output feedback cost to allstate feedback cost but was still limited to design for this specific initial condition. Levine[19] determined the

controller which was optimal in an average sense and computed time varying feedback gains for the case of the finite time-interval problem. For the semi-infinite time interval problem he required his feedback gains to be time-invariant. Cassidy[20] considered the minimization of a modified cost functional having the form

$$J = \frac{1}{2} \int_0^T (\underline{x}^T Q\underline{x} + \underline{x}^T \hat{Q}\underline{x} + \underline{x}^T W\underline{u} + \underline{x}^T \hat{W}\underline{y} + \underline{u}^T R\underline{u})\, d\tau \qquad (I.2\text{-}1)$$

where the matrices $\hat{Q}$ and $\hat{W}$ were chosen to ensure that the feedback gains for the unavailable states were zero. The resultant feedback gains were again time-varying for a finite time interval, time-invariant for the semi-infinite time interval.

It is considered that the approaches used by Levine and Cassidy are the most satisfactory of those considered, since they are both completely independent of initial conditions. They both, however, result in time-varying gains for the finite time-interval case. For the semi-infinite time interval both give constant feedback gains, but use computational algorithms which require initialization by a set of stabilizing feedback gains.

The problem of determining stabilizing output-feedback gains for complex systems has not, to the author's knowledge, been satisfactorily solved. Koenigsberg[21] has considered local stability maxima, but his computational algorithm is felt to be inefficient. Jameson[22] and Davison[23] have shown that as many system roots as there are feedback states can be arbitrarily determined, but the remaining roots are then unconstrained.

A solution to the problem of determining optimal constant or piecewise constant feedback gains for a finite time interval has been derived by

Kleinman, Fortmann and Athans.[25] As with Levine,[19] they considered optimality in an average sense, but confined attention to the allstate feedback case.

I.3 Scope and Contribution of This Work

This report considers the determination of optimal time-invariant output feedback control gains for systems which can be described by the equation

$$\dot{\underline{x}}(t) = A\,\underline{x}(t) + B\,\underline{u}(t) \qquad \text{(I.3-1)}$$

where $\underline{x}(t)$ is an NS-vector describing the state of the system at time t

$\underline{u}(t)$ is an NC-vector describing the control inputs at time t

A,B are time invariant matrices of appropriate order.

It is assumed, without loss of generality, that the system outputs are the first NF states. If this is not the case then a new set of state variables may be obtained as follows to satisfy the assumption. Suppose that the system described by (I.3-1) has NF independent outputs $\underline{y}(t)$ given by

$$\underline{y}(t) = C\,\underline{x}(t) + D\,\underline{u}(t) \qquad \text{(I.3-2)}$$

We shall require that

$$\underline{u}(t) = -K^T\,\underline{y}(t) \qquad \text{(I.3-3)}$$

where K is the time-invariant matrix of feedback gains. Thus

$$\left[I + DK^T\right]\,\underline{y}(t) = C\,\underline{x}(t) \qquad \text{(I.3-4)}$$

where I is the identity matrix of order NF. Assuming $\left[I + DK^T\right]$ to be non-singular gives

$$\underline{y}(t) = \hat{C}\,\underline{x}(t) \tag{I.3-5}$$

where

$$\hat{C} = \left[I + DK^T\right]^{-1} C \tag{I.3-6}$$

We note that in most cases D will be zero, leaving $\hat{C}$ equal to C. We now form the vector $\hat{\underline{x}}(t)$ comprising any (NS - NF) of the $\underline{x}(t)$ state variables which are independent of the $\underline{y}(t)$ variables. We note that such variables must exist since $\underline{x}(t)$ spans NS space while $\underline{y}(t)$ only spans NF space. Appending $\hat{\underline{x}}(t)$ to $\underline{y}(t)$ gives

$$\underline{z}(t) = \begin{bmatrix} \underline{y}(t) \\ \hat{\underline{x}}(t) \end{bmatrix} = \begin{bmatrix} \hat{C} \\ Ip \end{bmatrix} \underline{x}(t) \tag{I.3-7}$$

where Ip is a permutation matrix describing how $\hat{\underline{x}}(t)$ was chosen from $\underline{x}(t)$.

It follows that

$$\dot{\underline{z}}(t) = \begin{bmatrix} \hat{C} \\ Ip \end{bmatrix} \dot{\underline{x}}(t) \tag{I.3-8}$$

i.e.,

$$\dot{\underline{z}}(t) = \begin{bmatrix} C \\ Ip \end{bmatrix} A\,\underline{x}(t) + \begin{bmatrix} C \\ Ip \end{bmatrix} B\,\underline{u}(t) \tag{I.3-9}$$

Thus, from (I.3-7)

$$\dot{\underline{z}}(t) = \begin{bmatrix} C \\ Ip \end{bmatrix} A \begin{bmatrix} C \\ Ip \end{bmatrix}^{-1} \underline{z}(t) + \begin{bmatrix} C \\ Ip \end{bmatrix} B\,\underline{u}(t) \tag{I.3-10}$$

and $\underline{z}(t)$ is the desired state-vector having its first NF states as the system outputs.

Thus the control inputs $\underline{u}(t)$ for the system described by (I.3-1) are to be optimized over the set

$$\underline{u}(t) = - K^T \underline{x}(t) \tag{I.3-11}$$

where K, the time invariant matrix of feedback gains, is constrained to the form

$$K = \begin{bmatrix} \overbrace{k_{ij}}^{NC} \\ \hline 0 \end{bmatrix} \begin{matrix} \} \text{ NF} \\ \} \text{ NS} \end{matrix} \tag{I.3-12}$$

The cost functional is assumed to be of the form

$$J = \underline{x}^T(T)\, F\, \underline{x}(T) + \int_0^T \left[\underline{x}^T(\tau)\, Q \underline{x}(\tau) + \underline{u}^T(\tau)\, R\, \underline{u}(\tau) \right] d\tau \tag{I.3-13}$$

where F and Q are positive semi-definite

R is positive definite

and optimization is over the time interval $\left[0 , T \right]$.

Two separate cases are investigated. For the case where the terminal time T is infinite it is considered that the techniques presented by either Levine[19] or Cassidy[20] are adequate. Both these techniques, however, require initialization by stabilizing feedback gains. In practice, indeed the author has found that feedback gains giving marginal stability may be inadequate, due to numerical considerations. Chapter II of this report therefore presents a computational algorithm designed to find feedback gains of the form given by (I.3-12) such that the system stability is driven to a local maximum. Note that maximum stability is here defined to mean

that the least stable of the system roots is made as stable as possible. The algorithm presented is felt to be superior to that given by Koenigsberg[21] in that it is computationally faster.

For the case where the terminal time T is finite, an approach is taken similar to that used by Levine[19], except that the feedback gains are required to be time-invariant. The resulting theory is presented in Chapter III, together with a computational algorithm to allow the determination of the optimal gains.

It is felt that this report will help to fill some of the gaps in the existing ability to determine practical feedback controllers which are optimal in the engineering sense.

CHAPTER II

THE GAIN INITIALIZATION PROBLEM

II.1 Introduction

The problem of determination of the output feedback gains required to stabilize a dynamic system is an interesting one in its own right. In this report, however, we are concerned with the determination of such gains as a pre-requisite to the application of optimization techniques.

Both Levine[19] and Cassidy[20] have considered the determination of optimal output feedback controllers for linear systems. Both have derived iterative techniques resulting in time invariant feedback gains when the system is optimized over the semi-infinite time interval $[0, \infty]$. Each technique, however, requires initialization with a set of stabilizing feedback gains in order to ensure convergence of the computational algorithm.

Based on the author's practical experience[26] with Cassidy's technique it appears that an extra requirement on initial system stability may be added when the optimization algorithms are mechanized on a digital computer. It was found that with low order systems (five states or less) marginal stability was sufficient to ensure convergence of the optimization algorithm. When a twenty-one state model of the Saturn V booster rocket was considered, however, initialization by feedback gains giving marginal stability was insufficient to cause convergence. This was judged to be due to numerical truncation resulting from the finite word length in the digital computer. The problem was solved by computing a new set of feedback gains giving better than marginal stability. Re-initialization of the optimization algorithm with the new feedback gains resulted in convergence. It is considered likely that Levine's algorithm would have similar characteristics.

It is clear that practical application of Levine's and Cassidy's techniques requires a method for determination of stabilizing output feedback gains. It would be desirable that the gains give more than marginal stability. A method is described in this Chapter for determination of such gains, when they exist.

II.2 Theory of the Gain Initialization Technique

The problem under consideration is as follows. Given the system

$$\dot{\underline{x}}(t) = A\,\underline{x}(t) + B\,\underline{u}(t) \qquad \text{(II.2-1)}$$

where $\underline{x}(t)$ is an NS-vector describing the state of the system at time t

$\underline{u}(t)$ is an NC-vector describing the control inputs at time t

A,B are matrices whose elements depend on the characteristics of the system

with

$$\underline{u}(t) = -K^T\,\underline{x}(t) \qquad \text{(II.2-2)}$$

where K is a matrix of feedback gains such that

$$\dot{\underline{x}}(t) = \left[A - BK^T\right]\,\underline{x}(t) \qquad \text{(II.2-3)}$$

where K is constrained to the form

$$K = \overbrace{\left[\begin{array}{c} k_{ij} \\ \cdots\cdots \\ 0 \end{array}\right]}^{NC} \left.\begin{array}{c} \\ \\ \end{array}\right\} NF \left.\begin{array}{c} \\ \\ \end{array}\right\} NS \qquad \text{(II.2-4)}$$

how should the feedback gain matrix K be modified, within the constraints of (II.2-4) to increase the stability of the least stable eigenvalue of (II.2-3)

The solution hinges on an expression given by Fadeev and Fadeeva[27] for the sensitivity of an eigenvalue of a matrix to a parameter of the matrix. Given any eigenvalue λ and corresponding row and column eigenvectors $\underline{v}^T$ and $\underline{w}$ of a square matrix D, the sensitivity of the eigenvalue to a parameter α of the matrix is given by

$$\frac{d\lambda}{d\alpha} = \frac{\underline{v}^T \frac{\partial D}{\partial \alpha} \underline{w}}{\underline{v}^T \underline{w}} \tag{II.2-5}$$

We note that $\underline{v}$ and $\underline{w}$ satisfy

$$D^T \underline{v} = \lambda \underline{v} \tag{II.2-6}$$

$$D\underline{w} = \lambda \underline{w} \tag{II.2-7}$$

To apply (II.2-5) to the system described by (II.2-3) and (II.2-4) we note that

$$\frac{\partial [A-BK^T]}{\partial k_{pq}} = \underbrace{\left[0 \;\vdots\; \underline{b}_q \;\vdots\; 0 \right]}_{p^{th} \text{ column}} \tag{II.2-8}$$

where $\underline{b}_q$ is the q^{th} column of the matrix B. Thus if λ_i is any eigenvalue of (II.2-3) and if $\underline{v}_i^T$ and $\underline{w}_i$ are corresponding row and column eigenvectors we see that

$$\frac{\partial \lambda_i}{\partial k_{pq}} = \frac{\underline{v}_i^T \underbrace{\left[0 \;\vdots\; \underline{b}_q \;\vdots\; 0 \right]}_{p^{th} \text{ column}} \underline{w}_i}{\underline{v}_i^T \underline{w}_i} \tag{II.2-9}$$

The sensitivity of the real part of λ_i to the real feedback gain k_{pq} is simply the real part of the right hand side of (II.2-9).

Suppose now that it is desired to decrease the real part of λ_i. The direction of the steepest descent in feedback gain space is defined by the gradient matrix G where

$$g_{pq} = \text{Re}\left\{\frac{\partial \lambda_i}{\partial k_{pq}}\right\} \qquad \text{(II.2-10)}$$

$p = 1, \ldots, NF$

$q = 1, \ldots, NC$

II.3 The Gain Initialization Algorithm

A gain initialization algorithm has been derived from the expressions given in (II.2-9) and (II.2-10), and has been incorporated in the digital computer program GRADGN. A listing of the program GRADGN is given in Appendix I. A description of the algorithm/program follows.

The algorithm is iterative. It increases the stability of the least stable eigenvalue of the system at each step. If, during a given interation a new eigenvalue should become less stable than the one being operated on, this fact is taken into account in subsequent iterations.

Consider the rearrangement of the variable elements of the feedback gain matrix K into an NF x NC vector "K". The rearrangement is performed by column ordering of K so that

$$K_{pq} = {}^{"}K^{"}_{(q-1)NF + p} \qquad \text{(II.3-1)}$$

The gradient matrix G is similarly rearranged giving

$$G_{pq} = {}^{"}G^{"}_{(q-1)NF + p} \qquad \text{(II.3-2)}$$

The steepest descent method requires that variations in the gain vector $^{n}K^{n}$ must lie along the negative of the gradient vector $^{n}G^{n}$. Thus

$$^{n}\Delta K^{n} = -\mu\, ^{n}G^{n} \qquad \text{(II.3-3)}$$

where μ is a positive constant determined by the step size to be taken and $^{n}\Delta K^{n}$ is a variation in the gain vector. Suppose now that we wish to determine the variation $^{n}\Delta K^{n}$ in the gain vector $^{n}K^{n}$ required to cause a small negative change $\Delta\, \mathrm{Re}\{\lambda_i\}$ in the real part of the eigenvalue λ_i. The variation $^{n}\Delta K^{n}$ may be computed from the relationship

$$^{n}G^{nT}\, ^{n}\Delta K^{n} = \Delta\, \mathrm{Re}\{\lambda_i\} \qquad \text{(II.3-4)}$$

When the variation $^{n}\Delta K^{n}$ lies in the direction of steepest descent then, from (II.3-3) and (II.3-4)

$$-^{n}G^{nT}\ \ \mu\, ^{n}G^{n} = \Delta\, \mathrm{Re}\{\lambda_i\} \qquad \text{(II.3-5)}$$

whence

$$\mu = \frac{-\Delta\, \mathrm{Re}\{\lambda_i\}}{\left\| ^{n}G^{n} \right\|^{2}} \qquad \text{(II.3-6)}$$

and so

$$^{n}\Delta K^{n} = \frac{-\Delta\, \mathrm{Re}\{\lambda_i\}}{\left\| ^{n}G^{n} \right\|^{2}}\ \ ^{n}G^{n} \qquad \text{(II.3-7)}$$

When the variation $\Delta\, \mathrm{Re}\{\lambda_i\}$ is sufficiently small the system corresponding to the feedback gains $(^{n}K^{n} + ^{n}\Delta K^{n})$ should have an eigenvalue whose real part approximates $\mathrm{Re}\{\lambda_i\} + \Delta\, \mathrm{Re}\{\lambda_i\}$.

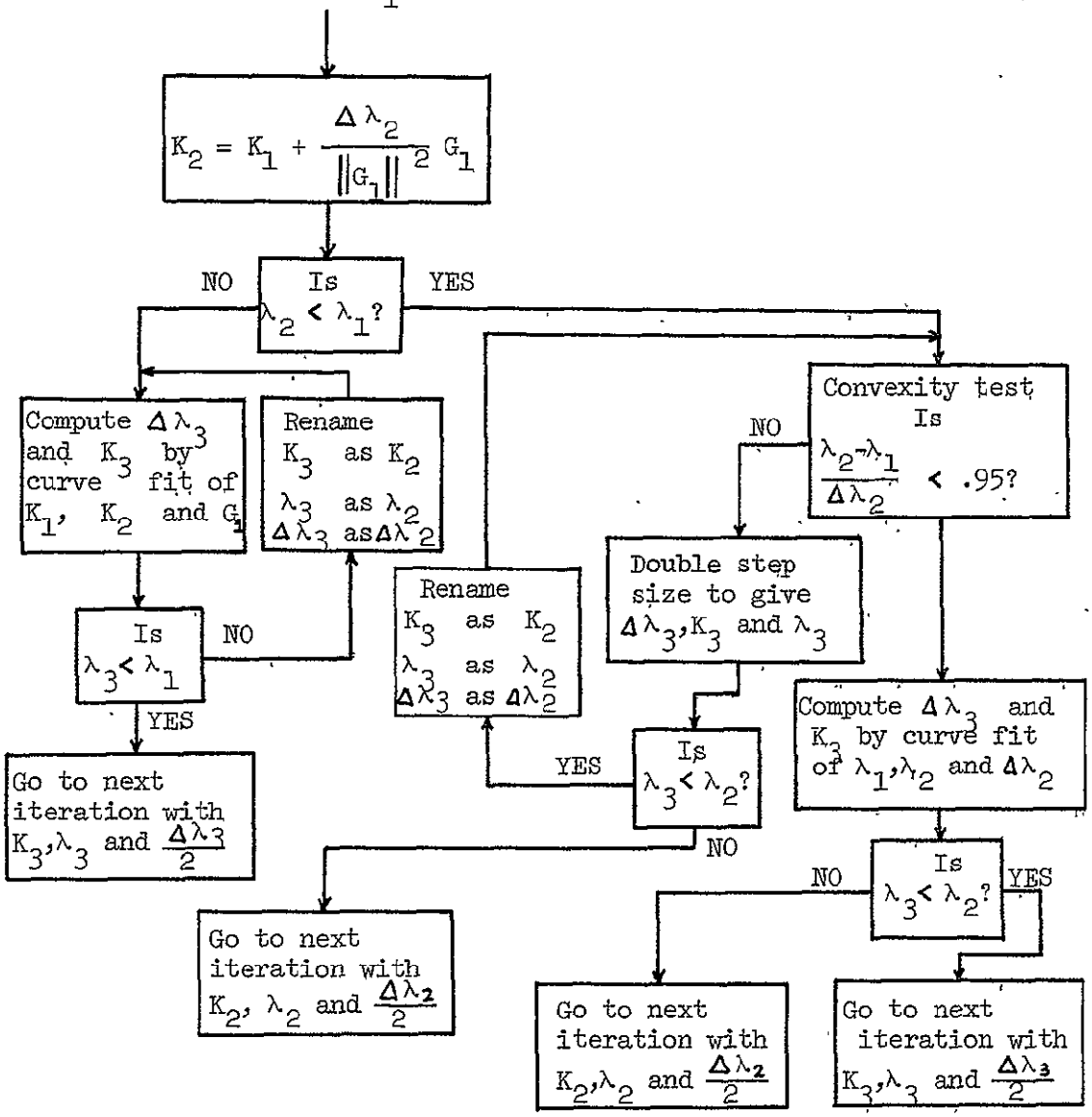
It remains to determine suitable values for the step size $\Delta \mathrm{Re}\{\lambda_i\}$. It is assumed that in most cases a user of GRADGN will have little idea of what comprises a suitable step size. The step size is therefore varied adaptively based on experience with previous iterations. The initial step size may either be assigned by the user or given a default value of -0.1. In either case the initial step size is used as a program criterion for termination due to diminishing returns. Should two successive iterations of the program fail to improve the stability by the initial step size, the program is terminated. It should be noted that apart from its use as a termination control the program is quite insensitive to the initial setp size, due to the rapidity of the step size adaptation.

The adaptation control for a given iterative step is described by Figure II.3-1. One of two basic step size adjustments is utilized, depending on the convexity of the trajectory of the least stable eigenvalue. If the trajectory is sufficiently convex then a quadratic curve is fitted to two points on the trajectory and the trajectory gradient at one of the points. This procedure is illustrated by Figure II.3-2. The gain vector corresponding to the minimum point of the quadratic curve is computed and tested for stability. If the trajectory is insufficiently convex then the step size is doubled. One or both of these procedures may be used repeatedly during a single iteration. Note that only one gradient vector is computed per iteration.

The initiation point for the next iteration is always the most stable gain vector found. The starting step size for the next iteration is set to one half of the current improvement in the least stable eigenvalue. This allows the step size to be changed by several orders of the magnitude in either direction during a single iteration, while allowing for the decreasing rate of improvement as a stability maximum is approached.

K_1, λ_1, $\Delta\lambda_2$ - from previous iteration

G_1 - computed from (II.2-9), (II.2-10) and (II.3-2)



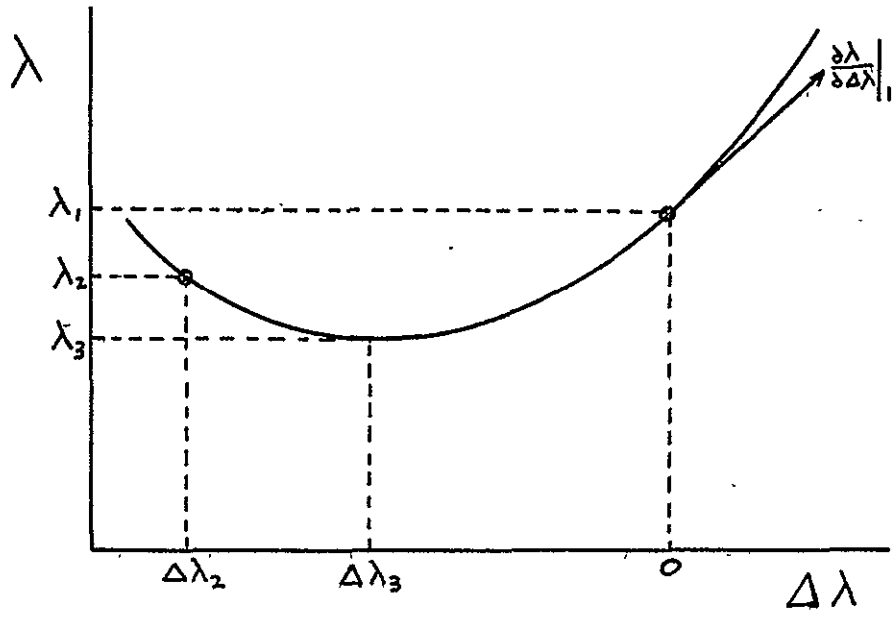
K_i = feedback gain vector at computation point i

λ_i = real part of least stable eigenvalue corresponding to feedback gains K_i

G_1 = gradient vector at computation point 1

$\Delta\lambda_i$ = step size from computation point 1 to computation point i

FIGURE II.3-1 Step Size Adaptation Control for the Gain Initialization Program GRADGN



$\Delta\lambda_i$ = step size from computation point 1 to computation point i

λ_i = real part of least stable eigenvalue at computation point i

Quadratic curve fit to variation of λ with $\Delta\lambda$ gives

$$\lambda = a(\Delta\lambda)^2 + b\Delta\lambda + c$$

and

$$c = \lambda_1$$

$$b = \frac{\partial\lambda}{\partial\Delta\lambda}\Big|_1 = 1$$

$$a = \frac{\lambda_2 - b\Delta\lambda_2 - c}{(\Delta\lambda_2)^2}$$

Curve has minimum point at $\Delta\lambda_3$ where $\frac{\partial\lambda}{\partial\Delta\lambda}\Big|_3 = 0$

$$\therefore\ \Delta\lambda_3 = -\frac{b}{2a} = -\frac{(\Delta\lambda_2)^2}{2(\lambda_2 - \lambda_1 - \Delta\lambda_2)}$$

FIGURE II.3-2 Step Size Adaptation in GRADGN by Quadratic Curve Fit

The computer program GRADGN incorporates an option for the degree of stability required for termination. Setting the input parameter ISTOP to zero instructs the program to maximize stability. Termination is then initiated by reaching a point of diminishing returns, as mentioned above. If ISTOP is set of 1 a minimum desired degree of stability, STOPR, may be input to the program. The program then terminates when the real part of the least stable eigenvalue becomes less than STOPR.

II.4 Example of the Use of the Gain Initialization Program GRADGN

The use of the gain initialization program GRADGN is illustrated in this Section by output feedback stabilization of a seven state model of the Saturn V booster rocket. The model represents the rocket dynamics at a point occuring 80 seconds after lift-off.

The seven states considered are

x_1 = measured pitch attitude angle ϕ_D

x_2 = measured pitch rate $\dot{\phi}_R$

x_3 = aerodynamic angle of attack α

x_4 = first bending mode deflection η_1

x_5 = first bending mode deflection rate $\dot{\eta}_1$

x_6 = engine gimbal angle β

x_7 = engine gimbal angle rate $\dot{\beta}$

It is assumed that only the first two states x_1 and x_2 are to be measured. Thus only these two states are available for feedback. Control is achieved via an actuator driving the engine gimbal angle at a rate proportional to the actuator input.

The model may be represented by

$$\dot{\underline{x}}(t) = A\,\underline{x}(t) + \underline{b}\,\underline{u}(t) \qquad \text{(II.4-1)}$$

with

$$\underline{u}(t) = -k_1\,x_1(t) - k_2\,x_2(t) \qquad \text{(II.4-2)}$$

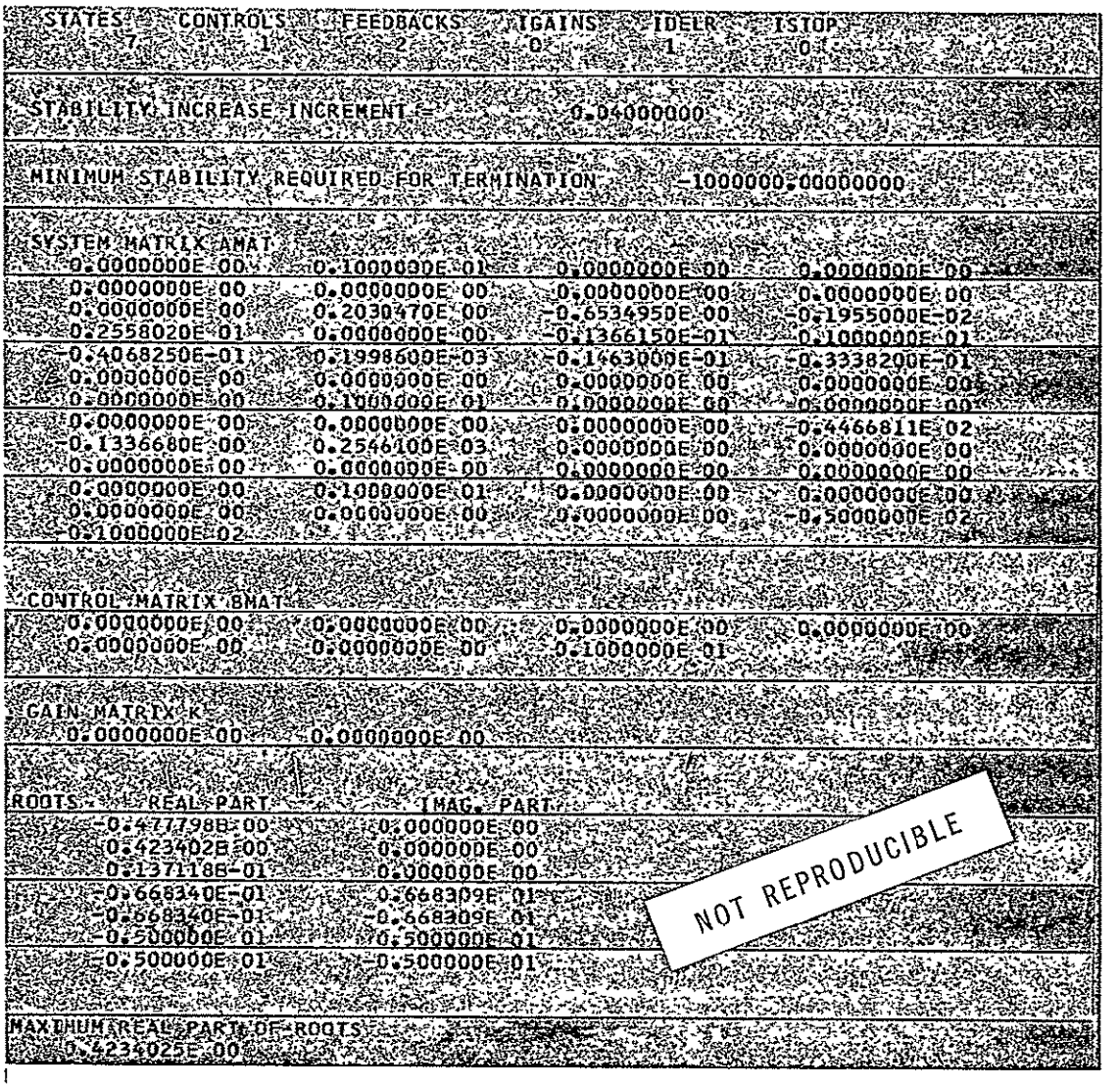
where

$$A = \begin{bmatrix} 0. & 1. & 0. & 0. & 0. & 0. & 0. \\ 0. & 0. & .2030 & -.6535 & -.0020 & 2.558 & 0. \\ -.0137 & 1. & -.0407 & .0002 & -.0146 & -.0334 & 0. \\ 0. & 0. & 0. & 0. & 1. & 0. & 0. \\ 0. & 0. & 0. & -44.67 & -.1337 & 254.6 & 0. \\ 0. & 0. & 0. & 0. & 0. & 0. & 1. \\ 0. & 0. & 0. & 0. & 0. & -50. & -10. \end{bmatrix} \qquad \text{(II.4-3)}$$

$$b = \begin{bmatrix} 0. \\ 0. \\ 0. \\ 0. \\ 0. \\ 0. \\ 1. \end{bmatrix} \qquad \text{(II.4-5)}$$

The derivation of this model is described by Cassidy.[20]

The program GRADGN was used to determine the feedback gains k_1 and k_2 corresponding to a local stability maximum of the above system. The program output is given in Figure II.4-1.



```
 STATES      CONTROLS     FEEDBACKS     IGAINS      IDELR      ISTOP
    7           1            2            0          1          0

 STABILITY INCREASE INCREMENT =          0.04000000

 MINIMUM STABILITY REQUIRED FOR TERMINATION      -1000000.00000000

 SYSTEM MATRIX AMAT
  0.00000000E 00   0.10000000E 01   0.00000000E 00   0.00000000E 00
  0.00000000E 00   0.00000000E 00   0.00000000E 00   0.00000000E 00
  0.00000000E 00   0.20304702E 00  -0.65349502E 00  -0.19550002E-02
  0.25580202E 01   0.00000000E 00  -0.13661502E-01   0.10000000E 01
 -0.40682502E-01   0.19986002E-03  -0.14630002E-01  -0.33382002E-01
  0.00000000E 00   0.00000000E 00   0.00000000E 00   0.00000000E 00
  0.00000000E 00   0.10000000E 01   0.00000000E 00   0.00000000E 00
  0.00000000E 00   0.00000000E 00   0.00000000E 00  -0.44668112E 02
 -0.13366802E 00   0.25461002E 03   0.00000000E 00   0.00000000E 00
  0.00000000E 00   0.00000000E 00   0.00000000E 00   0.00000000E 00
  0.00000000E 00   0.10000000E 01   0.00000000E 00   0.00000000E 00
  0.00000000E 00   0.00000000E 00   0.00000000E 00  -0.50000002E 02
 -0.10000000E 02

 CONTROL MATRIX BMAT
  0.00000000E 00   0.00000000E 00   0.00000000E 00   0.00000000E 00
  0.00000000E 00   0.00000000E 00   0.10000000E 01

 GAIN MATRIX K
  0.00000000E 00   0.00000000E 00

 ROOTS      REAL PART          IMAG. PART
 -0.47779882 00              0.00000000E 00
  0.42340282 00              0.00000000E 00
  0.13711882-01              0.00000000E 00
 -0.66834002E-01             0.66830992E 01
 -0.66834002E-01            -0.66830992E 01
 -0.50000002E 01             0.00000000E 01
 -0.50000002E 01             0.00000000E 01

 MAXIMUM REAL PART OF ROOTS
  0.42340252E 00
```


Figure II.4-1 Computer Print-Out For Example Problem

```
 GAIN SEARCH PROCEEDS ALONG A NEW GRADIENT. ITERATION NUMBER      1

 EIGVEC ERROR MESSAGES
 SW1=0.8345E-06          ITER=     4          DIF=0.3576E-06

 EIGENVECTORS CORRES TO MRP EIGENVALUE
    ROW REAL PART      ROW IMAG PART      COL REAL PART      COL IMAG PART
   -0.1411699E-01      0.0000000E 00      0.1000000E 01      0.0000000E 00
    0.1000000E 01      0.0000000E 00      0.4234034E 00      0.0000000E 00
    0.4375202E 00      0.0000000E 00     -0.8829014E 00      0.0000000E 00
    0.2059637E-03      0.0000000E 00     -0.2525228E-26      0.0000000E 00
   -0.1463000E-01      0.0000000E 00      0.8821868E-26      0.0000000E 00
   -0.2263338E 00      0.0000000E 00     -0.1349601E-27      0.0000000E 00
   -0.2171400E-01      0.0000000E 00     -0.4707256E-27      0.0000000E 00

 GRADIENT MATRIX
    0.2729351E-01      0.1155617E-01

 STABILITY INCREASE STEP SIZE =          0.04000000

 GAIN MATRIX K
   -0.1242759E 01     -0.5261886E 00

 ROOTS     REAL PART          IMAG. PART
      -0.708895E-01          0.666810E 01
      -0.708895E-01         -0.666810E 01
      -0.499273E 01          0.501011E 01
      -0.499273E 01         -0.501011E 01
      -0.452182E 00          0.000000E 00
       0.380997E 00          0.000000E 00
       0.239837E-01          0.000000E 00

 STEP SIZE IS DOUBLED

 GAIN MATRIX K
   -0.2485518E 01     -0.1052377E 01

 ROOTS     REAL PART          IMAG. PART
      -0.747366E-01          0.665300E 01
      -0.747366E-01         -0.665300E 01
      -0.498541E 01          0.502038E 01
      -0.498541E 01         -0.502038E 01
       0.424900E 00          0.000000E 00
       0.332093E 00          0.000000E 00
       0.389951E-01          0.000000E 00
```

FIGURE II.4-1 (Cont'd)

```
STEP SIZE IS DOUBLED

GAIN MATRIX K
  -0.4971038E 01     -0.2104753E 01

ROOTS      REAL PART          IMAG. PART
       -0.827146E-01       0.662188E 01
       -0.827146E-01      -0.662188E 01
       -0.497082E 01       0.504117E 01
       -0.497082E 01      -0.504117E 01
       -0.364258E 00       0.000000E 00
        0.148511E 00       0.256780E-01
        0.148511E 00      -0.256780E-01

STEP SIZE IS DOUBLED

GAIN MATRIX K
  -0.9942076E 01     -0.4209508E 01

ROOTS      REAL PART          IMAG. PART
       -0.979548E-01       0.655847E 01
       -0.979548E-01      -0.655847E 01
       -0.494167E 01       0.508360E 01
       -0.494167E 01      -0.508360E 01
        0.625687E-01       0.243981E 00
        0.625687E-01      -0.243981E 00
       -0.220044E 00       0.000000E 00

STEP SIZE IS DOUBLED

GAIN MATRIX K
  -0.1988414E 02     -0.8419017E 01

ROOTS      REAL PART          IMAG. PART
       -0.126488E 00       0.642529E 01
       -0.126488E 00      -0.642529E 01
       -0.488370E 01       0.517278E 01
       -0.488370E 01      -0.517278E 01
       -0.315896E-01       0.533504E 00
       -0.315896E-01      -0.533504E 00
       -0.907448E-01       0.000000E 00

STEP SIZE IS DOUBLED
```

FIGURE II.4-1 (Cont'd)

```
 GAIN MATRIX K
   -0.3976830E 02     -0.1683803E 02

ROOTS      REAL PART          IMAG. PART
        -0.477230E 01      -0.536959E 01
        -0.477230E 01       0.536959E 01
        -0.169915E 00       0.613263E 01
        -0.169915E 00      -0.613263E 01
        -0.113802E 00       0.913033E 00
        -0.113802E 00      -0.913033E 00
        -0.621070E-01       0.000000E 00

 STEP SIZE IS DOUBLED

 GAIN MATRIX K
   -0.7953661E 02     -0.3367606E 02

ROOTS      REAL PART          IMAG. PART
        -0.460448E 01       0.583911E 01
        -0.460448E 01      -0.583911E 01
        -0.144409E 00       0.542965E 01
        -0.144409E 00      -0.542965E 01
        -0.311907E 00       0.148304E 01
        -0.311907E 00      -0.148304E 01
        -0.528619E-01       0.000000E 00

 STEP SIZE TOO LARGE. COMPUTE NEW GRADIENT AT PREVIOUS BEST GAINS

 GAIN MATRIX K
   -0.3976830E 02     -0.1683803E 02

ROOTS      REAL PART          IMAG. PART
        -0.477230E 01       0.536959E 01
        -0.477230E 01      -0.536959E 01
        -0.169915E 00       0.613263E 01
        -0.169915E 00      -0.613263E 01
        -0.113802E 00       0.913033E 00
        -0.113802E 00      -0.913033E 00
        -0.621070E-01       0.000000E 00

 MAXIMUM REAL PART OF ROOTS
   -0.6210702E-01
```

FIGURE II.4-1 (Cont'd)

GAIN SEARCH PROCEEDS ALONG A NEW GRADIENT. ITERATION NUMBER 2

EIGVEC ERROR MESSAGES
SW1=0.1207E-02 ITER= 5 DIF=0.0000E 00

EIGENVECTORS CORRES TO MRP EIGENVALUE

ROW REAL PART	ROW IMAG PART	COL REAL PART	COL IMAG PART
0.1000000E 01	0.0000000E 00	0.2210334E 00	0.0000000E 00
0.1029001E 00	0.0000000E 00	-0.1373143E-01	0.0000000E 00
-0.9744528E 00	0.0000000E 00	0.1000000E 01	0.0000000E 00
-0.1416449E-01	0.0000000E 00	0.9880193E 00	0.0000000E 00
-0.1529490E-02	0.0000000E 00	-0.6137960E-01	0.0000000E 00
-0.1885111E-01	0.0000000E 00	0.1733184E 00	0.0000000E 00
-0.1896896E-02	0.0000000E 00	-0.1076721E-01	0.0000000E 00

GRADIENT MATRIX
-0.5431192E-03 0.3374061E-04

STABILITY INCREASE STEP SIZE = 0.24275460

GAIN MATRIX K
0.4054773E 03 -0.4449835E 02

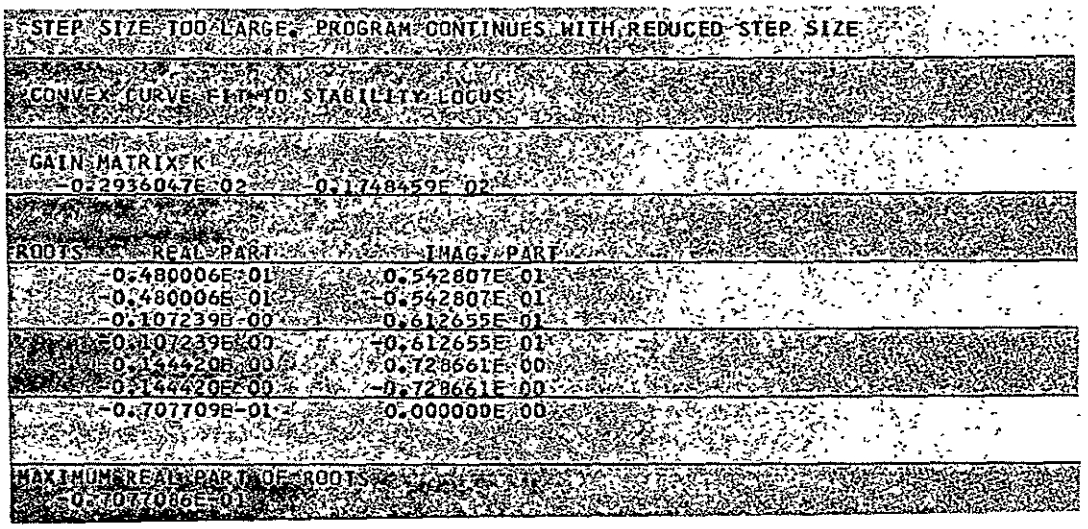
ROOTS REAL PART	IMAG. PART
-0.5869548 01	0.679102E 01
-0.586954E 01	-0.679102E 01
0.137404E 01	0.644784E 01
0.137404E 01	-0.644784E 01
-0.313356E 01	0.000000E 00
0.199437E 01	0.000000E 00
-0.442964E-01	0.000000E 00

CONVEX CURVE FIT TO STABILITY LOCUS

GAIN MATRIX K
-0.1626357E 02 -0.1829822E 02

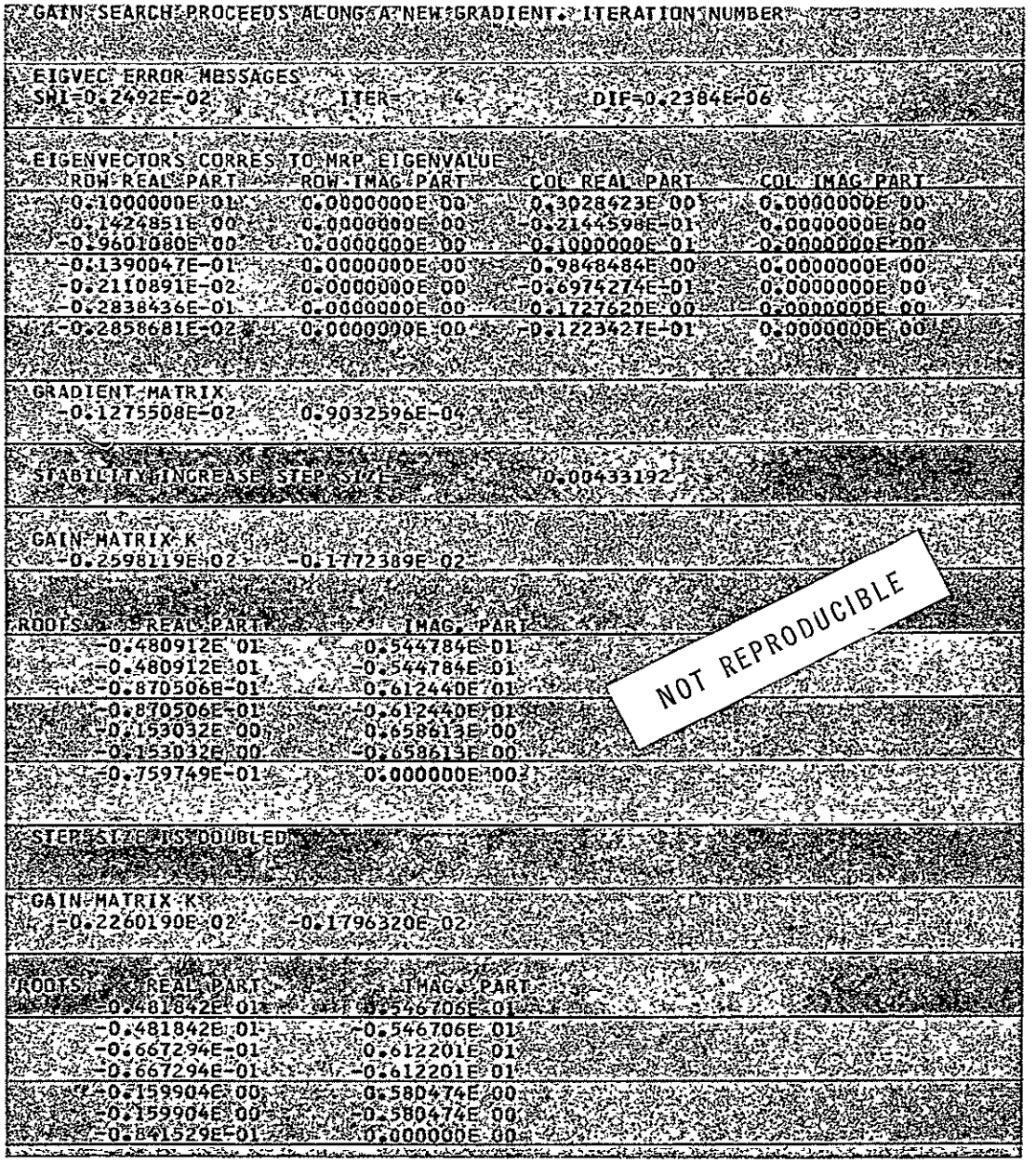
ROOTS REAL PART	IMAG. PART
-0.483643E 01	0.549917E 01
-0.483643E 01	-0.549917E 01
-0.307465E-01	0.612206E 01
-0.307465E-01	-0.612206E 01
-0.157808E 00	0.395643E 00
-0.157808E 00	-0.395643E 00
-0.124568E 00	0.000000E 00

FIGURE II.4-1 (Cont'd)



```
STEP SIZE TOO LARGE. PROGRAM CONTINUES WITH REDUCED STEP SIZE

CONVEX CURVE FIT TO STABILITY LOCUS

GAIN MATRIX K
   -0.2936047E 02      -0.1748459E 02

ROOTS      REAL PART              IMAG. PART
        -0.480006E 01          0.542807E 01
        -0.480006E 01         -0.542807E 01
        -0.107239E 00         -0.612655E 01
        -0.107239E 00          0.612655E 01
        -0.144420E 00          0.728661E 00
        -0.144420E 00         -0.728661E 00
        -0.707709E-01          0.000000E 00

MAXIMUM REAL PART OF ROOTS
   -0.7077086E-01
```

FIGURE II.4-1 (Cont'd)



```
GAIN SEARCH PROCEEDS ALONG A NEW GRADIENT. ITERATION NUMBER    3

 EIGVEC ERROR MESSAGES
 SNI=0.2492E-02          ITER=    4          DIF=0.2384E-06

 EIGENVECTORS CORRES TO MRP EIGENVALUE
    ROW REAL PART      ROW IMAG PART      COL REAL PART      COL IMAG PART
 0.1000000E 01      0.0000000E 00      0.3028423E 00      0.0000000E 00
 0.1424851E 00      0.0000000E 00     -0.2144598E-01      0.0000000E 00
-0.9601080E 00      0.0000000E 00      0.1000000E 01      0.0000000E 00
-0.1390047E-01      0.0000000E 00      0.9848484E 00      0.0000000E 00
-0.2110891E-02      0.0000000E 00     -0.6974274E-01      0.0000000E 00
-0.2838436E-01      0.0000000E 00     -0.1727620E 00      0.0000000E 00
-0.2858681E-02      0.0000000E 00     -0.1223427E-01      0.0000000E 00

 GRADIENT MATRIX
-0.1275508E-02      0.9032596E-04

 STABILITY INCREASE STEP SIZE=      0.00433192

 GAIN MATRIX K
-0.2598119E-02     -0.1772389E-02

 ROOTS      REAL PART          IMAG. PART
          -0.480912E 01      -0.544784E 01
          -0.480912E 01       0.544784E 01
          -0.870506E-01       0.612440E 01
          -0.870506E-01      -0.612440E 01
          -0.153032E 00       0.658613E 00
          -0.153032E 00      -0.658613E 00
          -0.759749E-01       0.000000E 00

 STEP SIZE HAS DOUBLED

 GAIN MATRIX K
-0.2260190E-02     -0.1796320E-02

 ROOTS      REAL PART          IMAG. PART
          -0.481842E 01       0.546706E 01
          -0.481842E 01      -0.546706E 01
          -0.667294E-01       0.612201E 01
          -0.667294E-01      -0.612201E 01
          -0.159904E 00       0.580474E 00
          -0.159904E 00      -0.580474E 00
          -0.841529E-01       0.000000E 00
```


FIGURE II.4-1 (Cont'd)

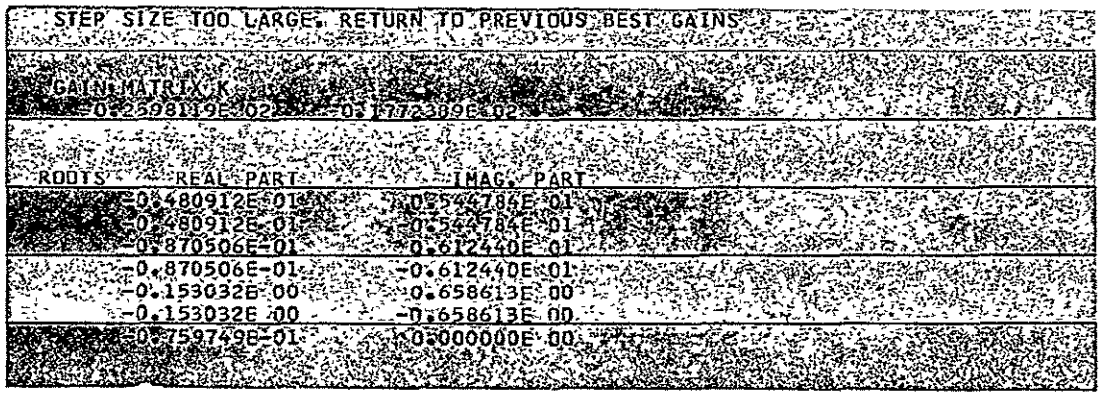


FIGURE II.4-1 (Cont'd)

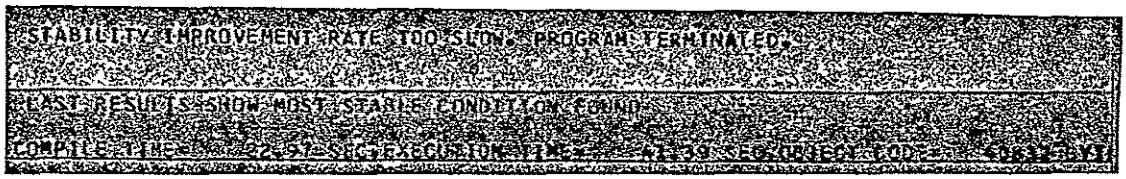


FIGURE II.4-1 (Cont'd)

In order to verify that GRADGN did indeed attain a stability maximum, the eigenvalues of the system described by (II.4-1) to (II.4-4) were obtained for a matrix of feedback gains. The values of the real parts of the least stable eigenvalues were cross-plotted against the feedback gains, resulting in the equi-stability contours shown in Figure II.4-2. The gain trajectory produced by GRADGN is superimposed on Figure II.4-2. The steepest descent nature of GRADGN, and the fact that a stability maximum was achieved (within the limits of the diminishing returns termination) are evident from Figure II.4-2.

II.5 Comments on Gain Initialization

The digital computer program GRADGN clearly provides a way to improve the stability of a system by output feedback. As such it is a useful adjunct to the output feedback optimization techniques of Cassidy and Levine. It must be remembered, however, that GRADGN is designed to find purely local stability maxima. Thus it may on occasion fail to find feedback gains giving sufficient stability for initialization of the optimization algorithms, even though such gains actually exist. In an attempt to ameliorate this problem, provision has been included in GRADGN for initializing its gain search at any desired location. Thus any desired volume of gain space may be searched by initializing GRADGN at a suitable number of discrete points.

Since GRADGN is designed to perform a similar function to Koenigsberg's[21] algorithm, a comparison of the two technique's may be pertinent. Both are gradient techniques and both use adaptive step size variation, although Koenigsberg's adaptation procedure is much simpler than that used in GRADGN. A major difference occurs in the computation of the gradient. Koenigsberg's

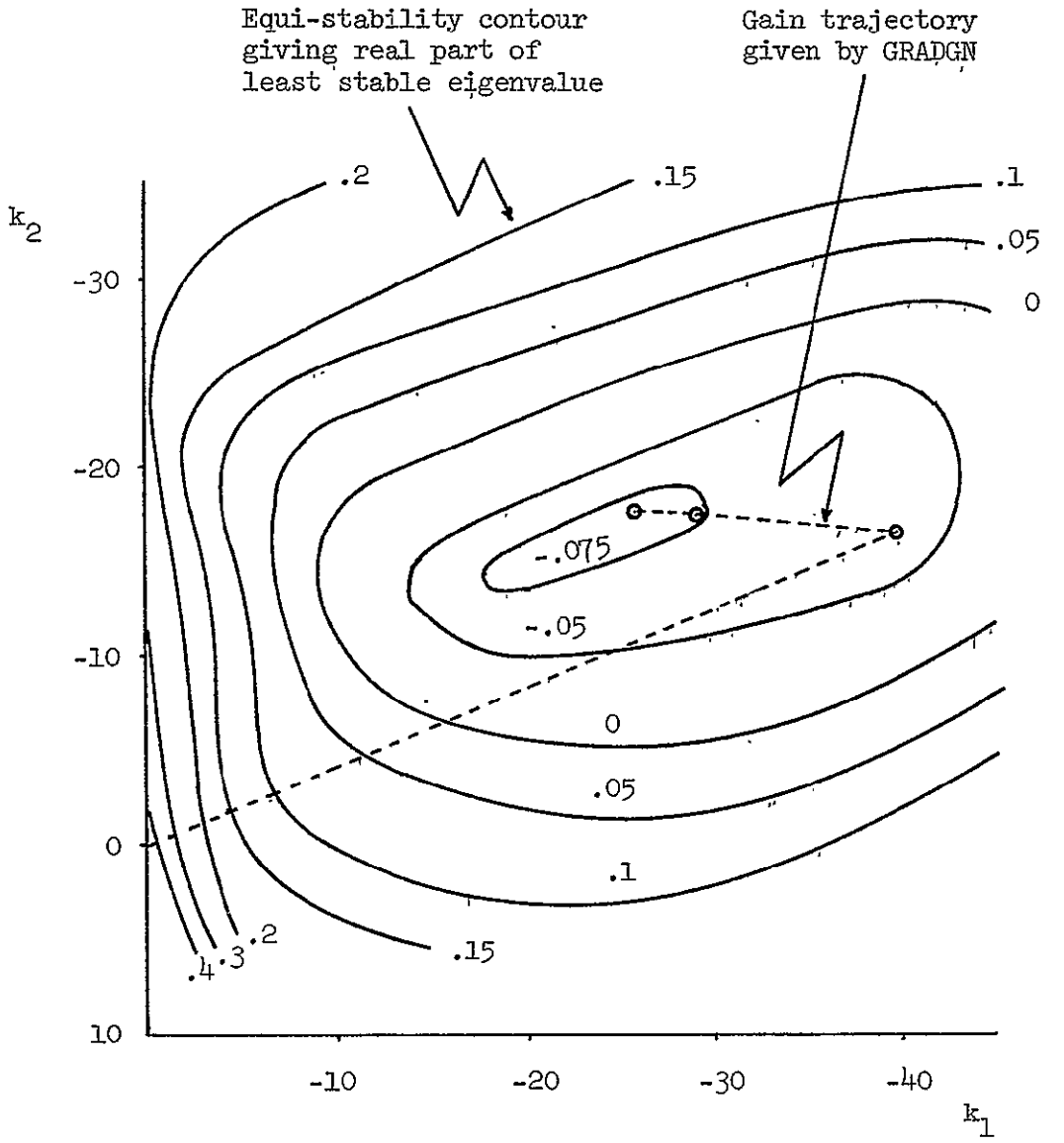


FIGURE II.4-2 Demonstration of GRADGN's Performance on Example Problem

gradient computation is based on Reddy's[30] expression

$$\frac{\partial \lambda_k}{\partial \alpha_p} = \left. \frac{\text{Tr (conjoint } [A] \left[\frac{\partial A}{\partial \alpha_p}\right])}{\text{Tr (conjoint } [A])} \right|_{\lambda = \lambda_k} \qquad \text{(II.5-1)}$$

for the sensitivity of the eigenvalue λ_k of the matrix A to the matrix parameter α_p. The conjoint is defined by

$$\text{conjoint } [A] = \text{adjoint } [A - \lambda I] \qquad \text{(II.5-2)}$$

The equivalent expression used by GRADGN is (from (II.2-5))

$$\frac{\partial \lambda_k}{\partial \alpha_p} = \frac{\underline{v}^T \frac{\partial A}{\partial \alpha_p} \underline{w}}{\underline{v}^T \underline{w}} \qquad \text{(II.5-3)}$$

where $\underline{v}$ and $\underline{w}$ respectively are the row and column eigenvectors of the matrix A corresponding to the eigenvalue λ_k. Equation (II.5-3) may be expanded to

$$\frac{\partial \lambda_k}{\partial \alpha_p} = \frac{\sum_{i=1}^{NS} \sum_{j=1}^{NS} v_i \left[\frac{\partial A}{\partial \alpha_p}\right]_{ij} w_j}{\sum_{i=1}^{NS} \sum_{j=1}^{NS} v_i \, w_j} \qquad \text{(II.5-4)}$$

which may be reordered to

$$\frac{\partial \lambda_k}{\partial \alpha_p} = \frac{\sum_{j=1}^{NS} \sum_{i=1}^{NS} w_j \, v_i \left[\frac{\partial A}{\partial \alpha_p}\right]_{ij}}{\sum_{j=1}^{NS} \sum_{i=1}^{NS} w_j \, v_i} \qquad \text{(II.5-5)}$$

Inspection of (II.5-5) shows it to be the same as

$$\frac{\partial \lambda_k}{\partial \alpha_p} = \frac{\mathrm{tr}\left[\underline{w}\,\underline{v}^T \frac{\partial A}{\partial \alpha_p}\right]}{\mathrm{tr}\left[\underline{w}\,\underline{v}^T\right]} \qquad \text{(II.5-6)}$$

Now the determination of a column eigenvector of a matrix may be achieved by the computation of only one column of the conjoint. Van Ness'[29] EIGVEC, used for eigenvector computation in GRADGN, gives both row and column eigenvectors with about the same computational effort as is required for only a column eigenvector. Thus (II.5-6) is computationally more efficient than (II.5-1). However (II.5-6) is simply (II.5-3) with each vector inner product replaced by the trace of a vector outer product, so that (II.5-3) is clearly preferable to (II.5-6) especially for systems of high order.

CHAPTER III

THE FINITE TIME INTERVAL PROBLEM

III.1 Introduction

In this Chapter we are concerned with the determination of the optimal time invariant output feedback controller for the system

$$\dot{\underline{x}}(t) = A\,\underline{x}(t) + B\,\underline{u}(t) \qquad \text{(III.1-1)}$$

where $\underline{x}(t)$ is a NS vector

$\underline{u}(t)$ is a NC vector

with the cost functional

$$J = \underline{x}^T(T)\,F\,\underline{x}(T) + \int_0^T \underline{x}^T(\tau)\,Q\,\underline{x}(\tau) + \underline{u}^T(\tau)\,R\,\underline{u}(\tau)\,d\tau \qquad \text{(III.1-2)}$$

where T is a fixed finite time and F, Q, R are suitably positive (semi) definite. Thus we require

$$\underline{u}(t) = -K^T\,\underline{x}(t). \qquad \text{(III.1-3)}$$

and constrain K to be of the form

$$K = \overbrace{\left[\begin{array}{c} k_{ij} \\ \hline 0 \end{array}\right]}^{NC} \begin{array}{l} \Big\} \text{ NF} \\ \Big\} \text{ NS} \end{array} \qquad \text{(III.1-4)}$$

Two items set this problem apart from the normal linear-quadratic state regulator problem. These are the requirement for output feedback control and the requirement for the time-invariant feedback gains. It is remarkable that each of these requirements has been individually satisfied by different investigators, but by the use of similar techniques. Levine[19]

used the Initial State Averaging (I.S.A.) technique to determine the optimal time-variable output feedback controller for the system (III.1-1) with the cost functional (III.1-2). On the other hand Kleinman, Fortmann and Athans[25] used I.S.A. to determine the optimal time-invariant allstate feedback controller. This report considers the use of I.S.A. to solve both requirements simultaneously.

The theory underlying the I.S.A. approach is presented in Section III.2

III.2 Theory of the Initial State Averaging Technique

The solution to the problem posed by (III.1-1) to (III.1-3) is underdefined in the sense that the optimal controller is a function of the undefined initial condition $\underline{x}(0)$. In general each different initial condition requires a different set of feedback gains. Since no single controller can minimize the cost functional for all initial conditions it seems wise to seek that controller which minimizes the expected value of the cost functional. This is the rationale behind I.S.A.

From (III.1-1) and (III.1-3) we see that

$$\dot{\underline{x}}(t) = \hat{A}\, \underline{x}(t) \tag{III.2-1}$$

where

$$\hat{A} = \left[A - BK^T \right] \tag{III,2-1}$$

Thus for a given initial condition $\underline{x}(0)$

$$\underline{x}(t) = \phi(t)\, \underline{x}(0) \tag{III.2-3}$$

where $\phi(t)$ is the state transition matrix of $\hat{A}$ and is defined by

$$\dot{\phi}(t) = \hat{A}\, \phi(t) \tag{III,2-4}$$

$$\phi(0) = I \tag{III.2-5}$$

Since we are dealing only with linear time-invariant systems we have[1]

$$\phi(t) = \exp\left[\hat{A}t\right] \tag{III.2-6}$$

Repeating (III.1-2)

$$J = \underline{x}^T(T)\ F\ \underline{x}(T) + \int_0^T \underline{x}^T(\tau)\ Q\ \underline{x}(\tau) + \underline{u}^T(\tau)\ R\ \underline{u}(\tau)\ d\tau \tag{III.2-7}$$

Using (III.1-3) reduces this to

$$J = \underline{x}^T(T)\ F\ \underline{x}(T) + \int_0^T x^T(\tau)\left[Q + KRK^T\right]\underline{x}(\tau)\ d\tau \tag{III.2-8}$$

whence, by (III.2-3)

$$J = \underline{x}(0)^T\ \phi^T(T)\ F\ \phi(T)\ \underline{x}(0) + \int_0^T \underline{x}^T(0)\ \phi^T(\tau)\left[Q + KRK^T\right]$$

$$\phi(\tau)\ \underline{x}(0)\ d\tau \tag{III.2-9}$$

Moving the time invariant $\underline{x}(0)$ outside the integration reduces (III.2-9) to

$$J = \underline{x}(0)^T\left[\phi^T(T)\ F\ \phi(T) + \int_0^T \phi^T(\tau)\left[Q + KRK^T\right]\phi(\tau)\ d\tau\right]\underline{x}(0) \tag{III.2-10}$$

which may be rewritten as

$$J = \underline{x}^T(0)\ P\ \underline{x}(0) \tag{III.2-11}$$

where

$$P = \phi^T(T)\ F\ \phi(T) + \int_0^T \phi^T(\tau) \left[Q + KRK^T \right] \phi(\tau)\ d\tau \qquad \text{(III.2-12)}$$

Thus

$$E\left[J\right] = E\left[x^T(0)\ P\ x\ (0) \right] \qquad \text{(III,2-13)}$$

i.e.,

$$E\left[J\right] = E\left[\sum_{i=1}^{NS} \sum_{j=1}^{NS} x_i(0)\ P_{ij}\ x_j(0) \right] \qquad \text{(III.2-14)}$$

Now consider the $x_i(0)$, $i = 1, \ldots, NS$ to be random variable whose distributions are dependent on the distribution of the initial conditions in state space. Then $E\left[J\right]$ is given by (III.2-14) as the expected value of a finite sum of random variables. But the expected value of a finite sum of random variables is equal to the sum of the expected values. Thus

$$E\left[J\right] = \sum_{i=1}^{NS} \sum_{j=1}^{NS} E\left[x_i(0)\ P_{ij}\ x_j(0) \right] \qquad \text{(III.2-15)}$$

Rearranging (III.2-15) gives

$$E\left[J\right] = \sum_{i=1}^{NS} \sum_{j=1}^{NS} E\left[P_{ij}\ x_i(0)\ x_j(0) \right] \qquad \text{(III.2-16)}$$

It is clear from (III.2-12) that the P_{ij} in (III.2-16) do not depend on the $x_i(0)$ or the $x_j(0)$. Thus (III.2-16) can be rewritten as

$$E\left[J\right] = \sum_{i=1}^{NS} \sum_{j=1}^{NS} P_{ij}\ E\left[x_j(0)\ x_i(0) \right] \qquad \text{(III.2-17)}$$

Suppose now that each of the $x_i(0)$ has zero mean and that we can estimate the covariance matrix V of the initial conditions in state space, i.e.,

$$V_{ij} = E\left[x_i(0)\ x_j(0)\right] \tag{III.2-18}$$

Then, from (III.2-17) and (III.2-18)

$$E\left[J\right] = \sum_{i=1}^{NS} \sum_{j=1}^{NS} P_{ij}\ V_{ij} \tag{III.2-19}$$

whence

$$E\left[J\right] = Tr\left[PV\right] \tag{III.2-20}$$

Now consider that the system described by (III.1-1), (III.1-3) and (III.1-4) is linear. Thus superposition holds and the system response can be scaled up or down with the magnitude of the initial conditions. This implies that the response of the system to an initial condition lying on the surface of the unit hypersphere in state space characterizes the response to any colinear initial condition state vector. Furthermore it is clear from (III.1-2) that if any initial condition is ratioed by an amount r then the cost is ratioed by r^2. These two facts imply that the costs associated with the set of initial conditions lying on the surface of the unit hypersphere is state space can be used to conveniently characterize the costs for all initial conditions. It is shown below that with little loss of generality attention may be confined to a uniform distribution of the initial conditions on the surface of the unit hypersphere.

Suppose that the covariance matrix V is non-singular with inverse V^{-1}. Suppose also that the square root of V^{-1} exists, and let it be denoted by $V^{-1/2}$. Consider the description of the system (III.2-1) in terms of a new set of state variables $\underline{y}(t)$ defined by

$$\underline{y}(t) = (NS)^{-1/2} \; V^{-1/2} \; \underline{x}(t) \tag{III.2-21}$$

Then (III.2-1) becomes

$$(NS)^{1/2} \; V^{1/2} \; \dot{\underline{y}}(t) = \hat{A}(NS)^{1/2} \; V^{1/2} \; \underline{y}(t) \tag{III.2-22}$$

i.e.,

$$\dot{\underline{y}}(t) = V^{-1/2} \; \hat{A} \; V^{1/2} \; \underline{y}(t) \tag{III.2-23}$$

which is similar in form to (III.2-1).

Also we find that

$$E\left[\underline{y}(0) \; \underline{y}^T(0)\right] = E\left[NS^{-1/2} \; V^{-1/2} \; \underline{x}(0) \; \underline{x}^T(0) \; (NS)^{-1/2} \; V^{-1/2}\right] \tag{III.2-24}$$

Thus

$$E\left[\underline{y}(0) \; \underline{y}^T(0)\right]_{ij} = (NS)^{-1} \; E\left[\sum_{p=1}^{NS} \; \sum_{q=1}^{NS} \; \left[V^{-1/2}\right]_{ip} \; x_p(0) \; x_q(0) \left[V^{-1/2}\right]_{qj}\right] \tag{III.2-25}$$

Interchanging finite sum and expectation gives

$$E\left[\underline{y}(0) \; \underline{y}^T(0)\right]_{ij} = (NS)^{-1} \; \sum_{p=1}^{NS} \; \sum_{q=1}^{NS} \; E\left[\left[V^{-1/2}\right]_{ip} \; x_p(0) \; x_q(0) \; \left[V^{-1/2}\right]_{qj}\right] \tag{III.2-26}$$

$$= (NS)^{-1} \; \sum_{p=1}^{NS} \; \sum_{q=1}^{NS} \; \left[V^{-1/2}\right]_{ip} \; V_{pq} \left[V^{-1/2}\right]_{qj} \tag{III.2-27}$$

$$= \frac{1}{NS} \left[V^{-1/2} \; V \; V^{-1/2} \right]_{ij} \qquad \text{(III.2-28)}$$

$$= \frac{1}{NS} \left[I \right]_{ij} \qquad \text{(III.2-29)}$$

where I is the identity matrix of order NS.

Thus the covariance matrix of the initial state vector $\underline{y}(0)$ is compatible with a uniform distribution of the $\underline{y}(0)$ vector on the surface of the unit hypersphere. Since such a distribution is convenient and since we can, within the limitations of the non-stringent assumptions made above, transform our initial state vector so that it is compatible with this distribution, we shall hereafter assume that

$$V = \frac{1}{NS} \; I \qquad \text{(III.2-30)}$$

Thus, from (III.2-20) we see that

$$E \left[J \right] = \frac{1}{NS} \; tr \left[P \right] \qquad \text{(III.2-31)}$$

and our I.S.A. optimal control is that which minimizes $\frac{1}{NS}$ tr $\left[P\right]$ where P is given by (III.2-12).

III.3 The Initial State Averaging Algorithm

The objective of the I.S.A. algorithm is to determine that gain matrix K which minimizes the average cost of control as given by (III.2-31) under the constraint that K must be of the form given by (III.1-4). Thus we wish to minimize the gain function GF given by

$$GF = \frac{1}{NS} \; tr \left[\phi^T(T) F \, \phi(T) + \int_0^T \phi^T(\tau) \left[Q + KRK^T \right] \phi(\tau) \, d\tau \right] \qquad \text{(III.3-1)}$$

Now for any two matrices A and B of suitable dimensions we have the trace identity

$$\text{tr}\left[AB\right] = \text{tr}\left[BA\right] \tag{III.3-2}$$

Using (III.3-2) and the fact that the integral of a matrix is the matrix of the integrals of the elements reduces (III.3-1) to

$$GF = \frac{1}{NS}\ \text{tr}\left[F\,\phi(T)\,\phi^T(T) + (Q + KRK^T)\int_0^T \phi(\tau)\,\phi^T(\tau)\,d\tau\right] \tag{III.3-3}$$

Since the variable elements of the gain matrix K are unconstrained the necessary conditions for K to minimize GF are given by

$$\frac{\partial GF}{\partial k_{ij}} = 0 \tag{III.3-4}$$

$i = 1, \ldots, NF$

$j = 1, \ldots, NC$

Expanding (III.3-4) gives the necessary conditions as

$$\text{tr}\left[F\left[\frac{\partial \phi}{\partial k_{ij}}\,\phi^T + \phi\frac{\partial \phi^T}{\partial k_{ij}}\right]_T + \left[\frac{\partial K}{\partial k_{ij}}\,R\,K^T + KR\,\frac{\partial K^T}{\partial k_{ij}}\right]\int_0^T \phi\,\phi^T\,d\tau\right.$$

$$\left. + \left[Q + KRK^T\right]\int_0^T \frac{\partial \phi}{\partial k_{ij}}\,\phi^T + \phi\,\frac{\partial \phi^T}{\partial k_{ij}}\,d\tau\right] = 0 \tag{III.3-5}$$

where the arguments of the ϕ functions have been dropped for clarity.

Now for any matrices A and B

$$\text{tr}\left[A\right] = \text{tr}\left[A^T\right] \tag{III.3-6}$$

and

$$\text{tr}\left[A + B\right] = \text{tr}\left[A\right] + \text{tr}\left[B\right] \qquad \text{(III.3-7)}$$

Thus in (III.3-5)

$$\text{tr}\left[F\left[\frac{\partial \phi}{\partial k_{ij}}\phi^T + \phi\frac{\partial \phi^T}{\partial k_{ij}}\right]\right] = \text{tr}\left[F\frac{\partial \phi}{\partial k_{ij}}\phi^T\right] + \text{tr}\left[F\,\phi\frac{\partial \phi^T}{\partial k_{ij}}\right]$$

by (III.3-7)

$$= \text{tr}\left[F\frac{\partial \phi}{\partial k_{ij}}\phi^T\right] + \text{tr}\left[\frac{\partial \phi}{\partial k_{ij}}\phi^T F\right]$$

by (III.3-6)

$$= \text{tr}\left[F\frac{\partial \phi}{\partial k_{ij}}\phi^T\right] + \text{tr}\left[F\frac{\partial \phi}{\partial k_{ij}}\phi^T\right]$$

by (III.3-2)

$$= 2\,\text{tr}\left[F\frac{\partial \phi}{\partial k_{ij}}\phi^T\right] \qquad \text{(III.3-8)}$$

Similarly

$$\text{tr}\left[\left[\frac{\partial K}{\partial k_{ij}}RK^T + KR\frac{\partial K^T}{\partial k_{ij}}\right]\int_0^T \phi\,\phi^T\,d\tau\right] = 2\,\text{tr}\left[\frac{\partial K}{\partial k_{ij}}RK^T\int_0^T \phi\,\phi^T\,d\tau\right]$$

(III.3-9)

and

$$\mathrm{tr}\left[\left[Q + KRK^T\right] \int_0^T \frac{\partial \phi}{\partial k_{ij}} \phi^T + \phi \frac{\partial \phi^T}{\partial k_{ij}} \, d\tau\right] = 2\,\mathrm{tr}\left[\left[Q + KRK^T\right] \int_0^T \phi \frac{\partial \phi^T}{\partial k_{ij}} \, d\tau\right] \tag{III.3-10}$$

Thus the necessary conditions of (III.3-5) reduce to

$$W = 0 \tag{III.3-11}$$

where the NF x NC matrix W has its elements defined by

$$w_{ij} = \mathrm{tr}\left[F \phi \frac{\partial \phi^T}{\partial k_{ij}}\bigg|_T + \left[Q + KRK^T\right] \int_0^T \phi \frac{\partial \phi^T}{\partial k_{ij}} \, d\tau + \frac{\partial K}{\partial k_{ij}} RK^T \int_0^T \phi \phi^T \, d\tau\right] \tag{III.3-12}$$

Equations (III.3-11) and (III.3-12) represent a set of NF x NC simultaneous equations in the NF x NC variable elements of the gain matrix K. Solution of these equations gives candidates for the otpimal controller. The solution is performed by Newton-Raphson iteration.

Consider the rearrangement of W into an NF x NC vector $"W"$ by column ordering

$$\text{i.e.,} \qquad w_{ij} = "W"_{NF(j-1)+i} \tag{III.3-13}$$

The variable elements of K may be similarly ordered, giving the gain vector $"K"$ where

$$K_{ij} = "K"_{NF(j-1)+i} \tag{III.3-14}$$

Suppose that $"K(n)"$ is the gain vector at the n^{th} iteration of the Newton-Raphson procedure, and that $"W(n)"$ is the corresponding

vector of necessary conditions. The Newton-Raphson technique computes $^{\square}K(n+1)^{\square}$, an improved estimate of a gain vector satisfying the necessary conditions, by

$$^{\square}K(n+1)^{\square} = {}^{\square}K(n)^{\square} - \mu \left[\nabla(n)\right]^{-1} {}^{\square}W(n)^{\square} \qquad \text{(III.3-15)}$$

where $\nabla(n)$ is the (NF x NC) x (NF x NC) gradient matrix given by

$$\nabla(n)_{ij} = \frac{\partial {}^{\square}W(n)^{\square}_i}{\partial {}^{\square}K(n)^{\square}_j} \qquad \text{(III.3-16)}$$

and μ is a convergence factor .

It can be seen from (III.3-13) and (III.3-14) that computation of $\nabla(n)$ is equivalent to computation of $\frac{\partial w_{gh}}{\partial k_{pq}}$ for $g = 1, \ldots, NF$, $h = 1, \ldots, NC$; $p = 1, \ldots, NF$; $q = 1, \ldots, NC$. From (III.3-12) we have

$$\begin{aligned}
\frac{\partial w_{gh}}{\partial k_{pq}} = \mathrm{tr}\Bigg[& F\left[\frac{\partial \phi}{\partial k_{pq}} \frac{\partial \phi^T}{\partial k_{gh}} + \phi \frac{\partial^2 \phi^T}{\partial k_{pq}\partial k_{gh}}\right]_T \\
& + \left[Q + KRK^T\right] \int_0^T \frac{\partial \phi}{\partial k_{pq}} \frac{\partial \phi^T}{\partial k_{gh}} + \phi \frac{\partial^2 \phi^T}{\partial k_{pq}\partial k_{gh}} \, d\tau \\
& + \left[\frac{\partial K}{\partial k_{pq}} RK^T + KR \frac{\partial K^T}{\partial k_{pq}}\right] \int_0^T \phi \frac{\partial \phi^T}{\partial k_{gh}} \, d\tau + \frac{\partial^2 K}{\partial k_{pq}\partial k_{gh}} RK^T \int_0^T \phi \, \phi^T \, d\tau \\
& + \frac{\partial K}{\partial k_{gh}} R\left[\frac{\partial K^T}{\partial k_{pq}} \int_0^T \phi \, \phi^T \, d\tau + K^T \int_0^T \frac{\partial \phi}{\partial k_{pq}} \phi^T + \phi \frac{\partial \phi^T}{\partial k_{pq}} d\tau\right]\Bigg]
\end{aligned} \qquad \text{(III.3-17)}$$

Clearly

$$\frac{\partial^2 K}{\partial k_{pq}\,\partial k_{gh}} = 0 \qquad \text{(III.3-18)}$$

Also, by (III.3-6) and (III.3-2)

$$\operatorname{tr}\left[KR \frac{\partial K^T}{\partial k_{pq}} \int_0^T \phi \frac{\partial \phi^T}{\partial k_{gh}}\, d\tau \right] = \operatorname{tr}\left[\int_0^T \frac{\partial \phi}{\partial k_{gh}} \phi^T\, d\tau \; \frac{\partial K}{\partial k_{pq}} RK^T \right]$$

$$= \operatorname{tr}\left[\frac{\partial K}{\partial k_{pq}} RK^T \int_0^T \frac{\partial \phi}{\partial k_{gh}} \phi^T\, d\tau \right] \qquad \text{(III.3-19)}$$

so that (III.3-17) reduces to

$$\frac{\partial w_{gh}}{\partial k_{pq}} = \operatorname{tr}\left[F \left[\frac{\partial \phi}{\partial k_{pq}} \frac{\partial \phi^T}{\partial k_{gh}} + \phi \frac{\partial^2 \phi^T}{\partial k_{pq}\,\partial k_{gh}} \right]_T \right.$$

$$+ \left[Q + KRK^T \right] \int_0^T \frac{\partial \phi}{\partial k_{pq}} \frac{\partial \phi^T}{\partial k_{gh}} + \phi \frac{\partial^2 \phi^T}{\partial k_{pq}\,\partial k_{gh}}\, d\tau$$

$$+ \frac{\partial K}{\partial k_{pq}} RK^T \int_0^T \frac{\partial \phi}{\partial k_{gh}} \phi^T + \phi \frac{\partial \phi^T}{\partial k_{gh}}\, d\tau$$

$$\left. + \frac{\partial K}{\partial k_{gh}} RK^T \int_0^T \frac{\partial \phi}{\partial k_{pq}} \phi^T + \phi \frac{\partial \phi^T}{\partial k_{pq}}\, d\tau + \frac{\partial K}{\partial k_{gh}} R \frac{\partial K^T}{\partial k_{pq}} \int_0^T \phi\, \phi^T\, d\tau \right]$$

(III.3-20)

From (III.3-3), (III.3-12) and (III.3-20) it can be seen that computation of the cost GF, the necessary conditions vector "W" and the gradient matrix ∇ for a given matrix K is dependent on the computation

of a number of functions of the state transition matrix $\phi(t)$. Basic to evaluation of these functions is evaluation of $\frac{\partial \phi(t)}{\partial k_{pq}}$ and $\frac{\partial^2 \phi(t)}{\partial k_{pq} \partial k_{gh}}$ for $p=1,\ldots,NF$; $q=1,\ldots,NC$; $g=1,\ldots,NF$; $h=1,\ldots,NC$.

Consider the state transition matrix $\phi_{\hat{K}}(t)$ corresponding to a gain matrix $\hat{K}$ given by

$$\hat{K} = K_0 + \frac{\partial K}{\partial k_{pq}} \delta k_{pq} + \frac{\partial K}{\partial k_{gh}} \delta k_{gh} \qquad \text{(III.3-21)}$$

where δk_{pq} and δk_{gh} are small. By (III.2-4) $\phi_{\hat{K}}(t)$ can be computed as the solution to

$$\frac{d\,\phi_{\hat{K}}(t)}{dt} = \left[A - BK_0^T - B\left(\frac{\partial K^T}{\partial k_{pq}} \delta k_{pq} + \frac{\partial K^T}{\partial k_{gh}} \delta k_{gh}\right)\right] \phi_{\hat{K}}(t) \qquad \text{(III.3-22)}$$

with

$$\phi_{\hat{K}}(0) = I \qquad \text{(III.3-23)}$$

Thus

$$\frac{d\,\phi_{\hat{K}}(t)}{dt} = \left[A-BK_0^T\right] \phi_{\hat{K}}(t) - B\left[\frac{\partial K^T}{\partial k_{pq}} \delta k_{pq} + \frac{\partial K^T}{\partial k_{gh}} \delta k_{gh}\right] \phi_{\hat{K}}(t) \qquad \text{(III.3-24)}$$

Treating the second term in (III.3-24) as an external input and using (III.2-6)

$$\phi_{\hat{K}}(t) = \exp\left[\left[A-BK_0^T\right] t\right] - \int_0^T \exp\left[\left[A-BK_0^T\right] (t-\tau)\right] B \left[\frac{\partial K^T}{\partial k_{pq}} \delta k_{pq} + \frac{\partial K^T}{\partial k_{gh}} \delta k_{gh}\right] \phi_{\hat{K}}(\tau)\, d\tau \qquad \text{(III.3-25)}$$

We now substitute the function described by (III.3-25) for $\phi_{\hat{K}}(\tau)$ in (III.3-25). This is similar to the method for computation of the matizant given in DeRusso, Roy and Close[1], and results in

$$\phi_{\hat{K}}(t) = \phi_{K_o}(t) - \int_0^t \phi_{K_o}(t-\tau)\, B \left[\frac{\partial K^T}{\partial k_{pq}} \delta k_{pq} + \frac{\partial K^T}{\partial k_{gh}} \delta k_{gh} \right]$$

$$\left[\phi_{K_o}(\tau) - \int_0^\tau \phi_{K_o}(\tau-s)\, B \left[\frac{\partial K^T}{\partial k_{pq}} \delta k_{pq} + \frac{\partial K^T}{\partial k_{gh}} \delta k_{gh} \right] \phi_{\hat{K}}(s)\, ds \right] d\tau$$

(III.3-26)

where $\phi_{K_o}(t)$ is the state transition matrix corresponding to the gain matrix K_o. Thus, to second order in δk_{pq} and δk_{gh},

$$\phi_{\hat{K}}(t) = \phi_{K_o}(t) - \int_0^t \phi_{K_o}(t-\tau)\, B \left[\frac{\partial K^T}{\partial k_{pq}} \delta k_{pq} + \frac{\partial K^T}{\partial k_{gh}} \delta k_{gh} \right]$$

$$\left[\phi_{K_o}(\tau) - \int_0^\tau \phi_{K_o}(\tau-s)\, B \left[\frac{\partial K^T}{\partial k_{pq}} \delta k_{pq} + \frac{\partial K^T}{\partial k_{gh}} \delta k_{gh} \right] \phi_{K_o}(s)\, ds \right] d\tau$$

(III.3-27)

Expansion of (III.3-27) gives

$$\phi_{\hat{K}}(t) = \phi_{K_o}(t) - \left[\int_0^t \phi_{K_o}(t-\tau)\, B \frac{\partial K^T}{\partial k_{pq}} \phi_{K_o}(\tau)\, d\tau \right] \delta k_{pq}$$

$$- \left[\int_0^t \phi_{K_o}(t-\tau)\, B \frac{\partial K^T}{\partial k_{gh}} \phi_{K_o}(\tau)\, d\tau \right] \delta k_{gh}$$

$$+\left[\int_0^t \phi_{K_0}(t-\tau)\, B\, \frac{\partial K^T}{\partial k_{pq}} \int_0^\tau \phi_{K_0}(\tau-s)\, B\, \frac{\partial K^T}{\partial k_{pq}}\, \phi_{K_0}(s)\, ds\, d\tau\right] \delta^2 k_{pq}$$

$$+\left[\int_0^t \phi_{K_0}(t-\tau)\, B\, \frac{\partial K^T}{\partial k_{pq}} \int_0^\tau \phi_{K_0}(\tau-s)\, B\, \frac{\partial K^T}{\partial k_{gh}}\, \phi_{K_0}(s)\, ds\, d\tau\right] \delta k_{pq}\, \delta k_{gh}$$

$$+\left[\int_0^t \phi_{K_0}(t-\tau)\, B\, \frac{\partial K^T}{\partial k_{gh}} \int_0^\tau \phi_{K_0}(\tau-s)\, B\, \frac{\partial K^T}{\partial k_{pq}}\, \phi_{K_0}(s)\, ds\, d\tau\right] \delta k_{pq}\, \delta k_{gh}$$

$$+\left[\int_0^t \phi_{K_0}(t-\tau)\, B\, \frac{\partial K^T}{\partial k_{gh}} \int_0^\tau \phi_{K_0}(\tau-s)\, B\, \frac{\partial K^T}{\partial k_{gh}}\, \phi_{K_0}(s)\, ds\, d\tau\right] \delta^2 k_{gh} \qquad \text{(III.3-28)}$$

Now consider that $\phi(t)$ can be expanded in a Taylor's series about K_0 giving

$$\phi_{\hat{K}}(t) = \phi_{K_0}(t) + \left.\frac{\partial \phi(t)}{\partial k_{pq}}\right|_{K_0} \delta k_{pq} + \left.\frac{\partial \phi(t)}{\partial k_{gh}}\right|_{K_0} \delta k_{gh}$$

$$+ \frac{1}{2!}\left[\left.\frac{\partial^2 \phi(t)}{\partial k_{pq}^2}\right|_{K_0} \delta^2 k_{pq} + 2\left.\frac{\partial^2 \phi(t)}{\partial k_{pq}\partial k_{gh}}\right|_{K_0} \delta k_{pq}\, \delta k_{gh}\right.$$

$$\left. + \left.\frac{\partial^2 \phi(t)}{\partial k_{gh}^2}\right|_{K_0} \delta^2 k_{gh}\right]$$

$$+ \text{terms of higher order} \qquad \text{(III.3-29)}$$

Comparison of (III.3-28) and (III.3-29) gives

$$\left.\frac{\partial \phi(t)}{\partial k_{pq}}\right|_{K_0} = - \int_0^t \phi_{K_0}(t-\tau)\, B\, \frac{\partial K^T}{\partial k_{pq}}\, \phi_{K_0}(\tau)\, d\tau \qquad \text{(III.3-30)}$$

and

$$\left.\frac{\partial^2 \phi(t)}{\partial k_{pq}\, k_{gh}}\right|_{K_0} = \int_0^t \phi_{K_0}(t-\tau)\, B\, \frac{\partial K^T}{\partial k_{pq}} \int_0^\tau \phi_{K_0}(\tau-s)\, \frac{\partial K^T}{\partial k_{gh}}\, \phi_{K_0}(s)\, ds\, d\tau$$

$$+ \int_0^t \phi_{K_0}(t-\tau)\, B\, \frac{\partial K^T}{\partial k_{gh}} \int_0^\tau \phi_{K_0}(\tau-s)\, \frac{\partial K^T}{\partial k_{pq}}\, \phi_{K_0}(s)\, ds\, d\tau$$

(III.3-31)

The result given by (III.3-30) agrees with a similar expression presented by Levine and Athans[31] for the semi-infinite time interval problem.

We note from (III.3-30) and (III.3-31) that

$$\left.\frac{\partial^2 \phi(t)}{\partial k_{pq} \partial k_{gh}}\right|_{K_0} = - \int_0^t \phi_{K_0}(t-\tau)\, B \left[\frac{\partial K^T}{\partial k_{pq}} \left.\frac{\partial \phi(t)}{\partial k_{gh}}\right|_{K_0} + \frac{\partial K^T}{\partial k_{gh}} \left.\frac{\partial \phi(t)}{\partial k_{pq}}\right|_{K_0} \right] d\tau$$

(III.3-32)

The functions of the state transition matrix to be evaluated are now seen to be given by

$$\phi \left.\frac{\partial \phi^T}{\partial k_{pq}}\right|_T = - \phi(T) \int_0^T \phi^T(\tau)\, \frac{\partial K}{\partial k_{pq}}\, B^T\, \phi^T(T-\tau)\, d\tau \qquad \text{(III.3-33)}$$

$$\frac{\partial \phi}{\partial k_{pq}} \left.\frac{\partial \phi^T}{\partial k_{gh}}\right|_T = \int_0^T \phi(T-\tau)\, B\, \frac{\partial K^T}{\partial k_{pq}}\, \phi(\tau)\, d\tau \int_0^T \phi^T(s)\, \frac{\partial K}{\partial k_{gh}}\, B^T \phi^T(T-s)\, ds$$

(III.3-34)

$$\phi \frac{\partial^2 \phi^T}{\partial k_{pq} \partial k_{gh}}\bigg|_T = -\phi(T) \int_0^T \left[\frac{\partial \phi^T(\tau)}{\partial k_{pq}} \frac{\partial K}{\partial k_{gh}} + \frac{\partial \phi^T(\tau)}{\partial k_{pq}} \frac{\partial K}{\partial k_{pq}}\right] B^T \phi(T-\tau) d\tau \tag{III.3-35}$$

$$\int_0^T \phi \frac{\partial \phi^T}{\partial k_{pq}} d\tau = - \int_0^T \phi(\tau) \int_0^\tau \phi^T(s) \frac{\partial K}{\partial k_{pq}} B^T \phi^T(\tau-s)\, ds\, d\tau \tag{III.3-36}$$

$$\int_0^T \frac{\partial \phi}{\partial k_{pq}} \frac{\partial \phi^T}{\partial k_{gh}} d\tau = \int_0^T \int_0^\tau \phi(\tau-\nu)\, B \frac{\partial K^T}{\partial k_{pq}} \phi(\nu)\, d\nu \tag{III.3-37}$$

$$\int_0^\tau \phi^T(s) \frac{\partial K}{\partial k_{gh}} B^T \phi^T(\tau-s)\, ds\, d\tau$$

and

$$\int_0^T \phi \frac{\partial^2 \phi^T}{\partial k_{pq} \partial k_{gh}} d\tau = - \int_0^T \phi(\tau) \int_0^\tau \left[\frac{\partial \phi(s)}{\partial k_{pq}} \frac{\partial k}{\partial k_{gh}} + \frac{\partial \phi^T(s)}{\partial k_{gh}} \frac{\partial K}{\partial k_{pq}}\right] B^T \phi(\tau-s) ds d\tau \tag{III.3-38}$$

Consider (III.3-33) as being representative of (III.3-33) through (III.3-38). Supposing $\hat{A}$ to have distinct eigenvalues we can evaluate $\phi(t)$ by

$$\phi(t) = M \exp\left[\Lambda t\right] M^{-1} \tag{III.3-39}$$

where M is the modal matrix of eigenvectors of $\hat{A}$

Λ is the diagonal matrix of eigenvalues of $\hat{A}$

Substitution of (III.3-39) in (III.3-33) yields

$$\phi \frac{\partial \phi^T}{\partial k_{pq}}\bigg|_T = - M e^{\Lambda T} M^{-1} \int_0^T \left[M^{-1}\right]^T e^{\Lambda \tau} M^T \frac{\partial K}{\partial k_{pq}} B^T \left[M^{-1}\right]^T e^{(T-\tau)} M^T d\tau \tag{III.3-40}$$

Now $\frac{\partial K}{\partial k_{pq}}$ is simply an NS x NC matrix having a 1.0 in it pq position and zeros elsewhere. Thus the ij element of $\phi \left.\frac{\partial \phi^T}{\partial k_{pq}}\right|_T$ is given by

$$\left[\phi \left.\frac{\partial \phi^T}{\partial k_{pq}}\right|_T\right]_{ij} = - \sum_{\substack{n_1,n_2,n_3 \\ n_4,n_5=1}}^{NS} M_{in_1} e^{\lambda n_1 T} M^{-1}_{n_1 n_2} \int_0^T M^{-1}_{n_3 n_2} e^{\lambda n_3} M_{pn_3} B_{n_4 q}$$

$$M^{-1}_{n_5 n_4} e^{\lambda n_5 (T-\tau)} M_{jn_5} \, d\tau \qquad \text{(III.3-41)}$$

where λ_{n_1} is the n_1^{th} eigenvalue of A.

Performing the integration yields

$$\left[\phi \left.\frac{\partial \phi^T}{\partial k_{pq}}\right|_T\right]_{ij} = - \sum_{\substack{n_1,n_2,n_3 \\ n_4,n_5=1}}^{NS} M_{in_1} M^{-1}_{n_1 n_2} M^{-1}_{n_3 n_2} M_{pn_3} B_{n_4 q} M^{-1}_{n_5 n_4} M_{jn_5}$$

$$\left[\underset{n_5=n_3}{\text{iff}} \; T e^{(\lambda n_1 + \lambda n_5)T} + \underset{n_5 \neq n_3}{\text{iff}} \; \frac{e^{(\lambda n_1 + \lambda n_3)T} - e^{(\lambda n_1 + \lambda n_5)T}}{\lambda n_3 - \lambda n_5} \right] \qquad \text{(III.3-42)}$$

Thus (III.3-33) can be evaluated in terms of the eigenvalues and eigenvectors of A. Equations (III.3-34) through (III.3-38) can be similarly evaluated. The terms can then be combined by (III.3-3), (III.3-12) and (III.3-20) to give the average cost GF, the necessary conditions vector $\|W\|$ and the gradient matrix ∇ respectively.

The only other item required for application of the Newton-Raphson iteration technique described by (III.3-15) is a value for the convergence

factor μ. Determination of a suitable value of the convergence factor is described in Section III.4.

III.4 Mechanization of the Initial State Averaging Algorithm

The algorithm described above for computation of optimal constant output feedback gains has been mechanized in the digital computer program ISAFT. The program was written in Fortran IV for an IBM 360/50 computer. A listing is given in Appendix II. The program comprises a MAIN and several subprograms. The functions of the major subprograms are described briefly below.

MAIN: performs most input-output functions and acts as the controller for the other subprograms. The MAIN subprogram also computes a suitable value for the Newton-Raphson convergence factor μ (see (III.3-15) and controls program termination. The operation is illustrated by the data flowchart given in Figure III.4-1.

Computation of the convergence factor μ is performed adaptively. A trial step is made with μ set to 1.0. The value of the gain function GF for the new gains is compared with its former value. If a decrease has occurred the program proceeds to the next iteration. Otherwise the value of μ is halved. This process is repeated up to five times. If no improvement in the gain function has resulted by the fifth cycle, i.e., $\mu = 2^{-5}$, the program is terminated due to poor convergence.

Assuming no termination due to poor convergence, normal program termination will occur when the feedback gains approach their final values. The criterion for this termination is that the proportional change in each individual gain during a single iteration should be less than an amount GNSTOP. The parameter GNSTOP is input by the user.

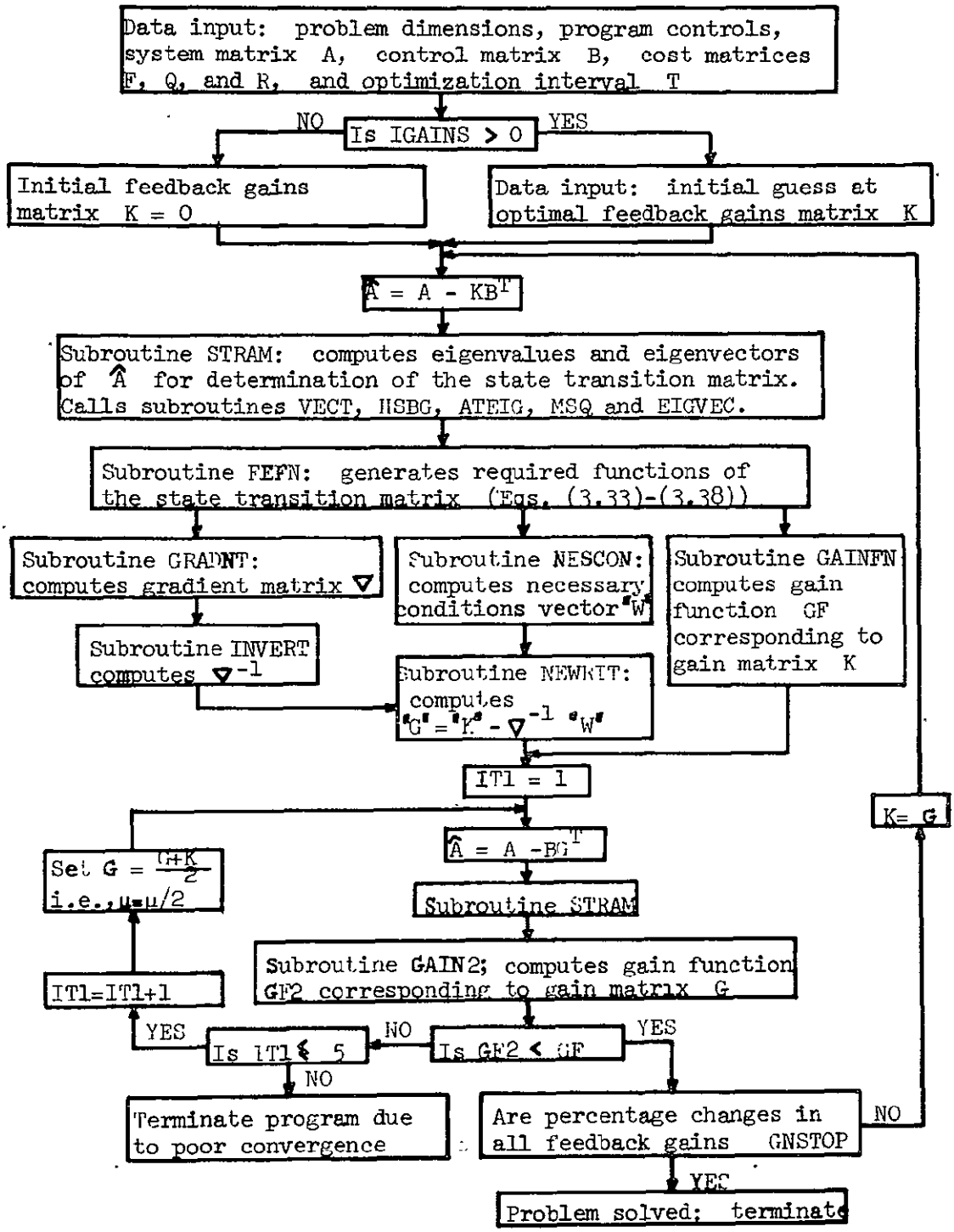


FIGURE III.4-1 Data Flowchart for Digital Computer Program ISAFT

STRAM: computes the eigenvalues and the model matrix of the matrix $\hat{A}$. The subprograms VECT, HSBG and ATEIG are used in the eigenvalue determination. The subprograms MSQ and EIGVEC are used to determine the model matrix and its inverse.

FEFN: computes the functions of the state transition matrix described by (III.3-33) through (III.3-38).

GAINFN: computes the value of the gain function GF at the beginning of each iteration - see (III.3-3).

NESCON: computes the necessary conditions vector "W" - see (III.3-12).

GRADNT: computes the gradient matrix ∇ - see (III.3-20).

INVERT: inverts the gradient matrix

NEWRIT: performs one iteration of the Newton-Raphson algorithm - see (III.3-15).

GAIN2: computes the value of the gain function GF at the end of each iteration for use in the determination of a suitable convergence factor μ - see (III.3-15).

III.5 Examples of the Use of the Optimization Program ISAFT

Three examples are given to illustrate the use of the digital computer program ISAFT. In keeping with the expository nature of this Chapter the examples are all short and are designed to illustrate different aspects of the computations.

Example 1

This example considered the case where all system states were available for feedback. The system equations were

$$\begin{bmatrix} \dot{x}_1(t) \\ \dot{x}_2(t) \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ -2 & -3 \end{bmatrix} \begin{bmatrix} x_1(t) \\ x_2(t) \end{bmatrix} + \begin{bmatrix} 0 \\ 1 \end{bmatrix} u(t) \qquad \text{(III.5-1)}$$

and the cost functional was

$$J = \underline{x}^T(1) \begin{bmatrix} 2 & 0 \\ 0 & 1 \end{bmatrix} \underline{x}(1) + \int_0^1 \underline{x}^T(\tau) \begin{bmatrix} 1 & 0 \\ 0 & 2 \end{bmatrix} \underline{x}(\tau) + 0.1\, u^2(\tau)\, d\tau \qquad \text{(III.5-2)}$$

The control was constrained to be of the form

$$u(t) = -k_1\, x_1(t) - k_2 x_2(t) \qquad \text{(III.5-3)}$$

It was assumed that there was no prior knowledge about the system, and the initial feedback gains were set to the default value of zero. The computer print out is given in Figure III.5-1. The optimal feedback gains were computed to be

$$k_1 = 2.30 \qquad \text{(III.5-4)}$$

$$k_2 = 3.42 \qquad \text{(III.5-5)}$$

To verify that the values given by (III.5-4) and (III.5-5) were indeed optimal the results were checked by independent means. A matrix of feedback gains in the vicinity of the solution gains was chosen. At each set of gains the system described by (III.5-1) and (III.5-3) was simulated over the optimization time interval, starting from a number of different initial conditions, and the costs were computed numerically using (III.5-2). The average cost was then computed for each set of feedback gains. The initial conditions were chosen to be equally spaced around the unit circle in state space. Such an independent check will hereafter be referred to as an I.C. Cross-plotting the I.C. data resulted in the equal-cost contours shown in Figure III.5-2. The gain trajectory produced by ISAFT is superimposed

```
 STATES    CONTROLS    FEEDBACKS    IGAINS
      2           1            2         0

 SYSTEM MATRIX A
      0.0000000        1.0000000       -2.0000000       -3.0000000

 CONTROL MATRIX B
      0.0000000        1.0000000

 TERMINAL TIME T =      1.0000000

 TERMINAL COST MATRIX F
      2.0000000        0.0000000        0.0000000        1.0000000

 STATE WEIGHTING MATRIX Q
      1.0000000        0.0000000        0.0000000        2.0000000

 CONTROL WEIGHTING MATRIX R
      0.1000000
```

FIGURE III.5-1 Computer Print-Out For Example 1

```
ITERATION NUMBER          1

GAIN MATRIX
        0.0000000          0.0000000

 SYSTEM EIGENVALUES
   -0.2000000D 01     -0.0000000D 00
   -0.1000000D 01     -0.0000000D 00

AVERAGE COST =         0.1227681D 01

NECESSARY CONDITIONS VECTOR
   -0.6159624D-01     -0.9331765D-01

 GRADIENT MATRIX
    0.2239452D 00     -0.9323973D-01     -0.9323973D-01      0.1338320D 00

 INVERSE GRADIENT MATRIX
        8.7898689          4.3821046          4.3821046         10.5250361
```

FIGURE III.5-1 (Cont'd)

```
ITERATION NUMBER          2

NEW GAINS
       0.7963600          1.2520928

SYSTEM EIGENVALUES
  -0.3438549D 01     -0.0000000D 00
  -0.8131438D 00     -0.0000000D 00

    GAIN TOLERANCE ACHIEVED =          1.0000000

REQUIRED STOPPING TOLERANCE =          0.0100000

AVERAGE COST =        0.1121653D 01

NECESSARY CONDITIONS VECTOR
  -0.2673182D-01     -0.3301758D-01

GRADIENT MATRIX
   0.1144905D 00    -0.5721144D-01    -0.5721144D-01     0.6673589D-01

INVERSE GRADIENT MATRIX
      15.2801933         13.0994265         13.0994265         26.2143374
```

FIGURE III.5-1 (Cont'd)

```
ITERATION NUMBER          3

NEW GAINS
       1.6373387          2.4677983

SYSTEM EIGENVALUES
  -C.46926910 01     -C.0CCCCCOD 00
  -C.7751C72D C0     -C.0CCCCCOD 00

     GAIN TOLERANCE ACHIEVED =          0.5136254

REQUIRED STOPPING TOLERANCE =          0.0100000

AVERAGE COST =        C.10836380 01

NECESSARY CONDITIONS VECTOR
  -C.8372973D-C2     -0.9586460D-02

GRADIENT MATRIX
   C.6909313D-01     -0.3735754D-01     -0.3735754D-01      0.3910556D-01

INVERSE GRADIENT MATRIX
      29.9352104         28.5971025         28.5971025         52.8906197
```

FIGURE III.5-1 (Cont'd)

```
ITERATION NUMBER

NEW GAINS
       2.1621304          3.2142749

SYSTEM EIGENVALUES
  -C.5450676D C1     -C.0C0C0000D 00
  -C.7635989D C0.    -C.CCCCCCOD 00

     GAIN TOLERANCE ACHIEVED =          0.2427197

REQUIRED STOPPING TOLERANCE =          0.0100000

AVERAGE COST =        C.1C771720 01

NECESSARY CONDITIONS VECTOR
  -C.1332E13D-C2     -C.1736675D-02

GRADIENT MATRIX
   C.5362835D-C1     -0.2976007D-01     -0.2976007D-01      0.2943540D-01

INVERSE GRADIENT MATRIX
      42.4808458         42.9494143         42.9494143         77.3958539
```

FIGURE III.5-1 (Cont'd)

```
ITERATION NUMBER          5

 NEW GAINS
        2.2933385          3.4059298

 SYSTEM EIGENVALUES
   -C.5645432D C1     -C.0C00000D 00
   -C.7604978D C0     -C.0C00000D 00

      GAIN TOLERANCE ACHIEVED =          0.0572127

 REQUIRED STOPPING TOLERANCE =           0.0100000

 AVERAGE COST =        0.1076910D 01

 NECESSARY CONDITIONS VECTOR
   -C.4792496D-04     -C.8692727D-04

 GRADIENT MATRIX
    C.5059286D-C1     -0.2817238D-01     -0.2817238D-01      0.2745821D-01

 INVERSE GRADIENT MATRIX
        46.1089873          47.3082380          47.3082380          84.9576547
```

FIGURE III.5-1 (Cont'd)

```
ITERATION NUMBER          6

 NEW GAINS
        2.2996607          3.4155822

  SYSTEM EIGENVALUES
    -0.5655292D 01     -0.0000000D 00
    -0.7602897D 00     -0.0000000D 00

      GAIN TOLERANCE ACHIEVED =          0.0028260

 REQUIRED STOPPING TOLERANCE =          0.0100000

  SOLUTION IS COMPLETE. ABOVE GAINS ARE OPTIMAL

 AVERAGE COST =        0.1076909D 01

COMPILE TIME=     28.40 SEC,EXECUTION TIME=    193.64 SEC,OBJECT CODE=    64192 BYTES,
```

FIGURE III.5-1 (Cont'd)

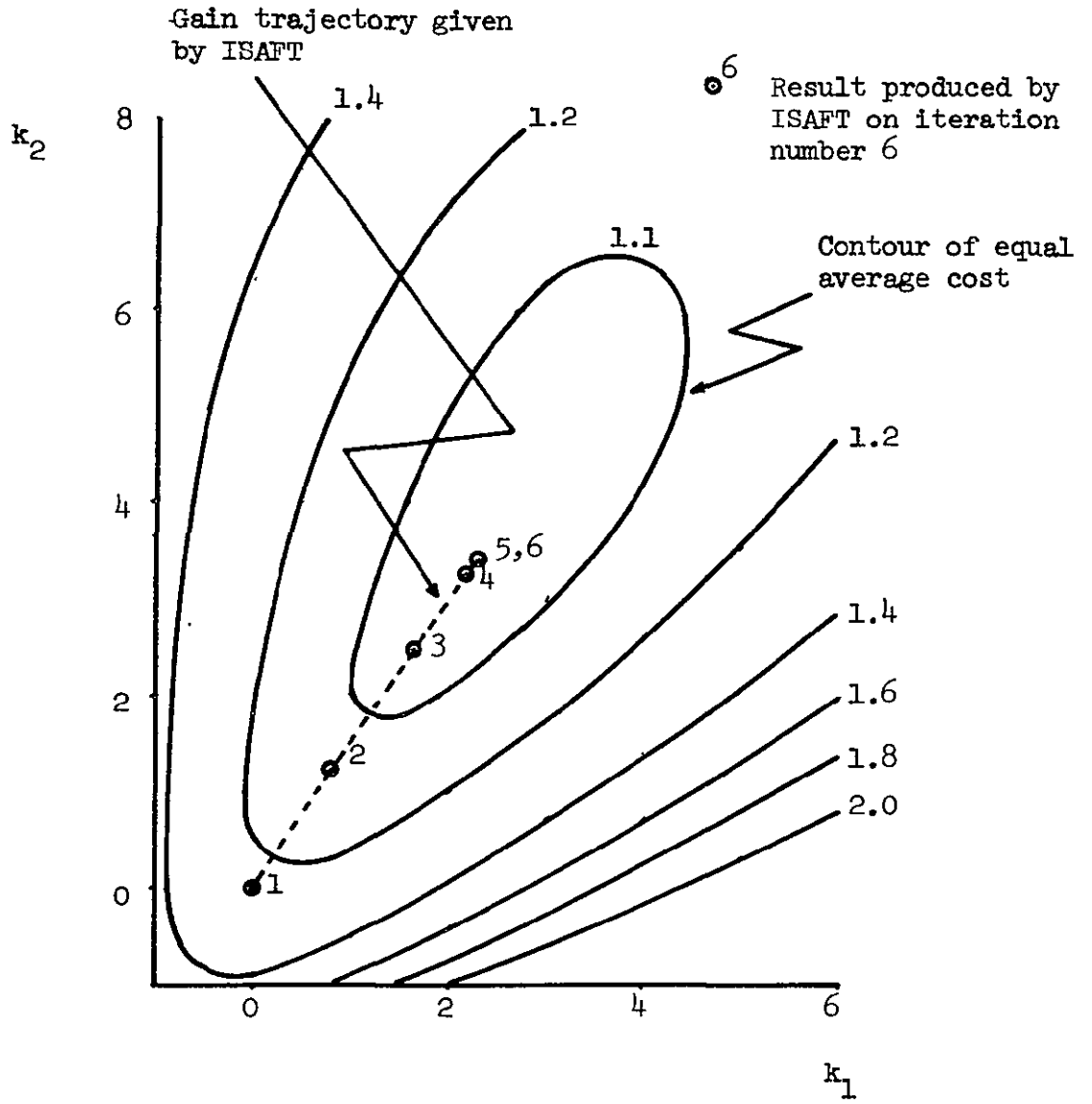


FIGURE III.5-2 Verification of the ISAFT results for Example 1

on Figure III.5-2. It is apparent that ISAFT determined feedback gains giving a minimum average cost.

To provide a comparison for the results produced by ISAFT, the problem defined by (III.5-1) and (III.5-2) was solved for the optimal time varying feedback gains by the normal procedure of Ricatti matrix backwards integration.[2] The resulting feedback gains are compared in Figure III.5-3. with those produced by ISAFT. The average cost for the optimal time varying gains was computed by numerical integration from a number of initial conditions. The average cost for the time-varying gains was 1.030 compared with 1.077 for the time invariant gains.

Example 2

This example used the same system equations and cost functional as Example 1 - see (III.5-1) and (III.5-2) respectively. The feedback structure however, was constrained to feedback of only the first state i.e.,

$$u(t) = - k_1 x_1(t) \qquad \text{(III.5-6)}$$

A rather inaccurate guess at the optimal feedback gain resulted in the computer print out given in Figure III.5-4. The use of the convergence factor is evident.

Figure III.5-5 compares the results given by ISAFT with those determined by an I.C. It is again clear that ISAFT determined a feedback gain giving a minimum average cost.

The minimum average cost for the single time-invariant feedback was 1.219 compared with 1.077 for allstate time invariant feedback and 1.030 for allstate time varying feedback.

Example 3

This example was chosen to check the effect of complex eigenvalues

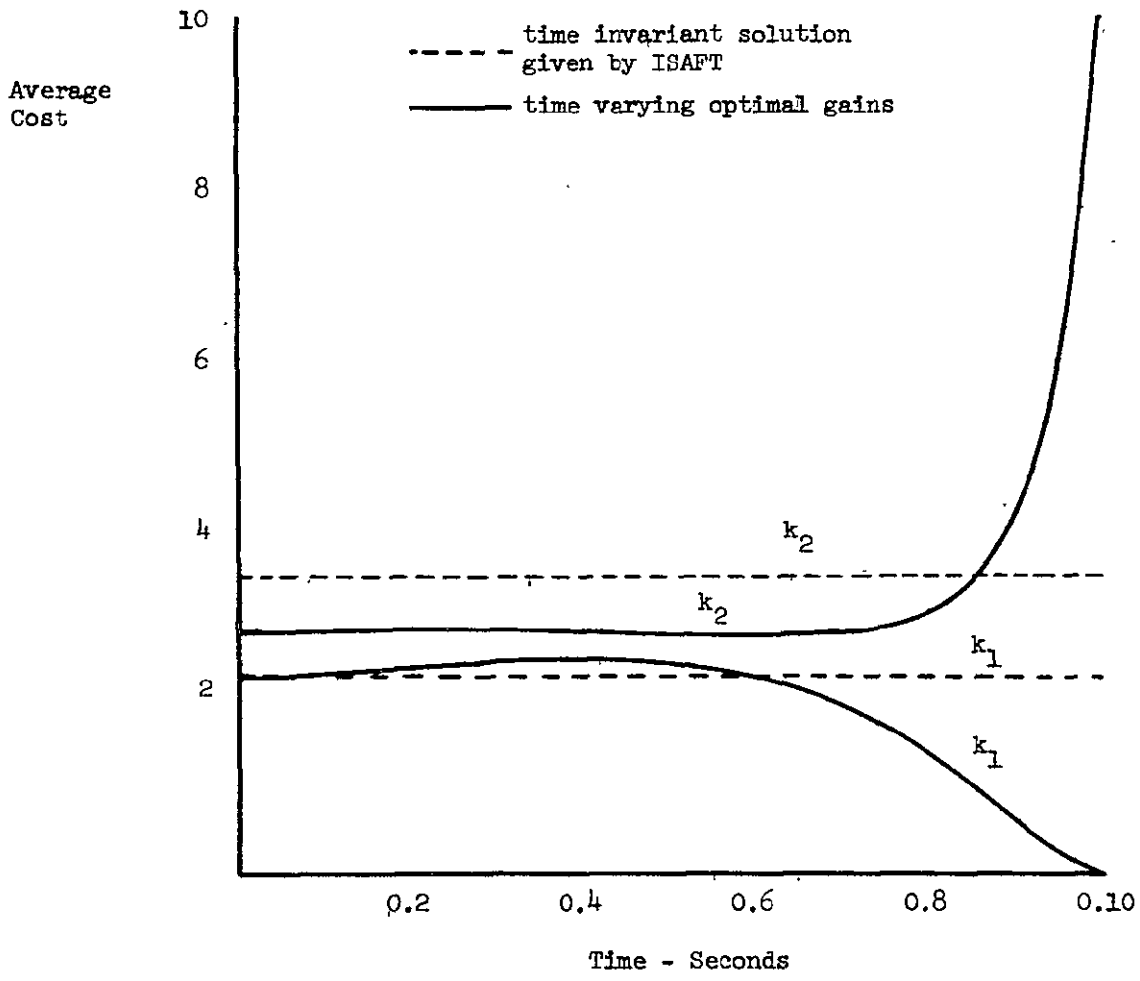


FIGURE III.5-3 Comparison of Time-varying and Time-invariant Optimal Feedback Gains for Example 1

STATES	CONTROLS	FEEDBACKS	IGAINS
2	1	1	1

SYSTEM MATRIX A

0.0000000	1.0000000	-2.0000000	-3.0000000

CONTROL MATRIX B

0.0000000	1.0000000

TERMINAL TIME T = 1.0000000

TERMINAL COST MATRIX F

2.0000000	0.0000000	0.0000000	1.0000000

STATE WEIGHTING MATRIX Q

1.0000000	0.0000000	0.0000000	2.0000000

CONTROL WEIGHTING MATRIX R

0.1000000

```
ITERATION NUMBER          1

 GAIN MATRIX
        5.0000000

 SYSTEM EIGENVALUES
   -0.1500000D 01      0.2179449D 01
   -0.1500000D 01     -0.2179449D 01

 AVERAGE COST =        0.2160457D 01

 NECESSARY CONDITIONS VECTOR
    0.2729108D 00

 GRADIENT MATRIX
    0.9892897D-02

 INVERSE GRADIENT MATRIX
      101.0826292
```

FIGURE III.5-4 (Cont'd)

```
ITERATION NUMBER        2

 NEW GAINS
     -22.5865428

 SYSTEM EIGENVALUES
   -0.6278760D 01     -0.0000000D 00
    0.3278760D 01      0.0000000D 00

 STEP SIZE TOO LARGE. NEW AVERAGE COST WOULD HAVE BEEN   0.3733863D 04
 GAIN ADJUSTMENT IS HALVED

 NEW GAINS
     -8.7932714

 SYSTEM EIGENVALUES
   -0.4507203D 01     -0.0000000D 00
    0.1507203D 01      0.0000000D 00

 STEP SIZE TOO LARGE. NEW AVERAGE COST WOULD HAVE BEEN   0.5075050D 02
 GAIN ADJUSTMENT IS HALVED

 NEW GAINS
     -1.8966357

 SYSTEM EIGENVALUES
   -0.2965140D 01     -0.0000000D 00
   -0.3485984D-01     -0.0000000D 00

    GAIN TOLERANCE ACHIEVED =          3.6362469

 REQUIRED STOPPING TOLERANCE =          0.0100000

 AVERAGE COST =          0.1924687D 01

 NECESSARY CONDITIONS VECTOR
   -0.7873214D 00

 GRADIENT MATRIX
    0.5870270D 00
```

FIGURE III.5-4 (Cont'd)

INVERSE GRADIENT MATRIX
1.7034993

FIGURE III.5-4 (Cont'd)

```
ITERATION NUMBER          3

 NEW GAINS
       -0.5554343

 SYSTEM EIGENVALUES
   -0.2397460D 01     -0.0000000D 00
   -0.6025401D 00     -0.0000000D 00

    GAIN TOLERANCE ACHIEVED =      2.4146897

 REQUIRED STOPPING TOLERANCE =      0.0100000

 AVERAGE COST =      0.1300264D 01

 NECESSARY CONDITIONS VECTOR
    -0.2070736D 00

 GRADIENT MATRIX
    0.3030369D 00

 INVERSE GRADIENT MATRIX
       3.2999278
```

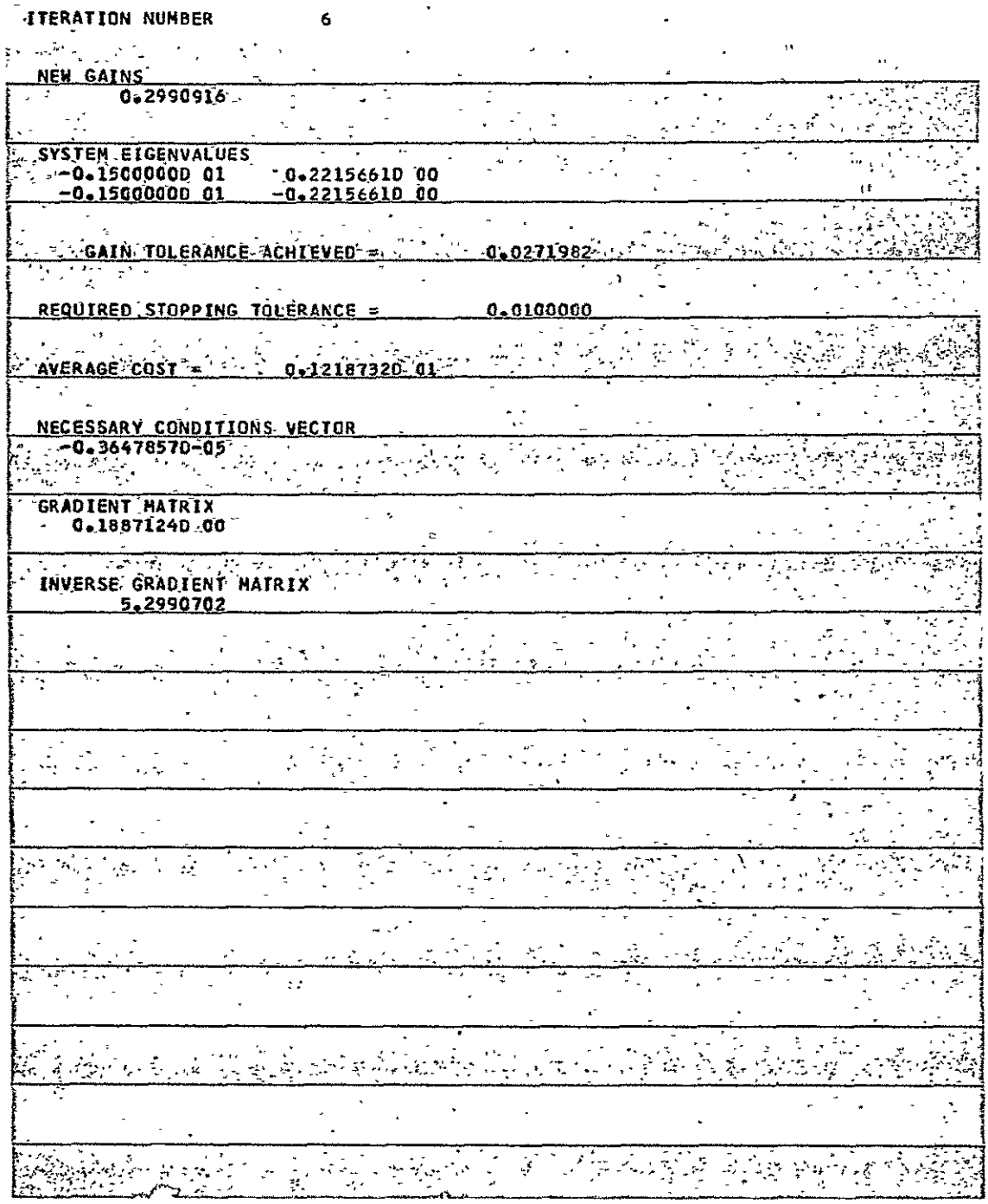
FIGURE III.5-4 (Cont'd)

```
ITERATION NUMBER

 NEW GAINS
        0.1278936

 SYSTEM EIGENVALUES
   -0.18494370 01     -0.00000000 00
   -0.11505630 01     -0.00000000 00

     GAIN TOLERANCE ACHIEVED =          5.3429401

 REQUIRED STOPPING TOLERANCE =          0.0100000

  AVERAGE COST =        0.12215920 01

 NECESSARY CONDITIONS VECTOR
   -0.3396536D-01

 GRADIENT MATRIX
    0.20829570 00

 INVERSE GRADIENT MATRIX
        4.8008683
```

FIGURE III.5-4 (Cont'd)

```
ITERATION NUMBER          5

 NEW GAINS
      0.2909568

 SYSTEM EIGENVALUES
   -0.15000000 01      0.2023779D 00
   -0.15000000 01     -0.2023779D 00

    GAIN TOLERANCE ACHIEVED =          0.5604378

 REQUIRED STOPPING TOLERANCE =         0.0100000

 AVERAGE COST =        0.1218738D 01

 NECESSARY CONDITIONS VECTOR
   -0.1542418D-02

 GRADIENT MATRIX
    0.1896087D 00

 INVERSE GRADIENT MATRIX
      5.2740210
```

FIGURE III.5-4 (Cont'd)



```
 ITERATION NUMBER          6

 NEW GAINS
          0.2990916

 SYSTEM EIGENVALUES
    -0.1500000D 01     0.2215661D 00
    -0.1500000D 01    -0.2215661D 00

      GAIN TOLERANCE ACHIEVED =          0.0271982

 REQUIRED STOPPING TOLERANCE =           0.0100000

 AVERAGE COST =       0.1218732D 01

 NECESSARY CONDITIONS VECTOR
    -0.3647857D-05

 GRADIENT MATRIX
     0.1887124D 00

 INVERSE GRADIENT MATRIX
          5.2990702
```

FIGURE III.5-4 (Cont'd)

```
ITERATION NUMBER        7

NEW GAINS
        0.2991109

 SYSTEM EIGENVALUES
   -0.15000000 01      0.2216098D 00
   -0.15000000 01     -0.2216098D 00

     GAIN TOLERANCE ACHIEVED =          0.0000646

 REQUIRED STOPPING TOLERANCE =          0.0100000

 SOLUTION IS COMPLETE. ABOVE GAINS ARE OPTIMAL

 AVERAGE COST =          0.1218732D 01

COMPILE TIME=     27.14 SEC,EXECUTION TIME=     69.23 SEC,OBJECT CODE=     64216 BYTES
```

FIGURE III.5-4 (Cont'd)

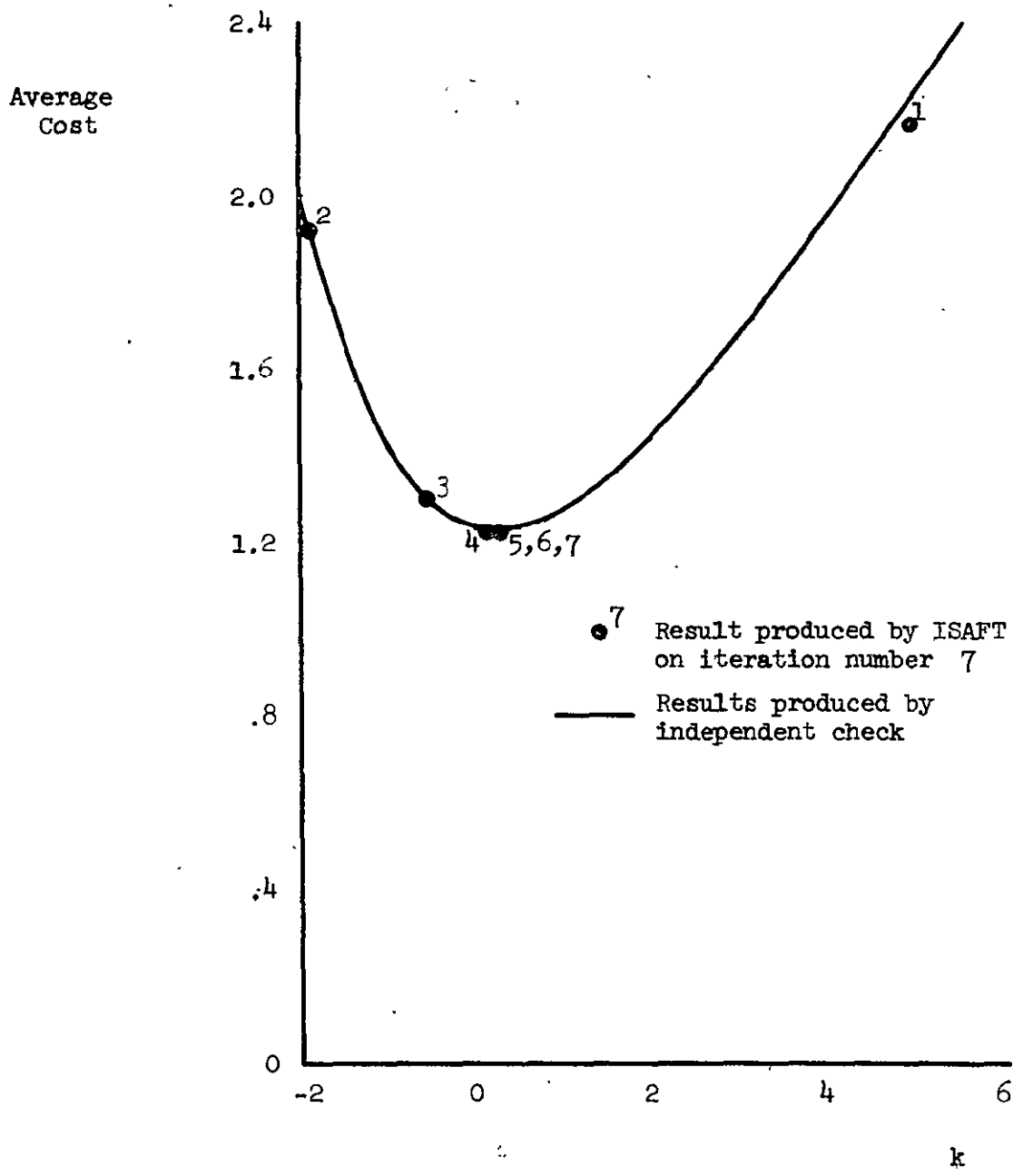


FIGURE III.5-5 Verification of the ISAFT results for Example 2

on the computations. The system equations were

$$\begin{bmatrix} \dot{x}_1(t) \\ \dot{x}_2(t) \\ \dot{x}_3(t) \end{bmatrix} = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 6 & 7 & 0 \end{bmatrix} \begin{bmatrix} x_1(t) \\ x_2(t) \\ x_3(t) \end{bmatrix} + \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} u(t) \qquad \text{(III.5-7)}$$

with the cost functional

$$J = \underline{x}^T(1) \begin{bmatrix} 3 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 1 \end{bmatrix} \underline{x}(1) + \int_0^1 \underline{x}^T(\tau) \begin{bmatrix} 1 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 3 \end{bmatrix} \underline{x}(\tau) + 0.1\, u^2(\tau)\, d\tau \qquad \text{(III.5-8)}$$

It was assumed that only the first state was available for feedback so that

$$u(t) = - k_1\, x_1(t) \qquad \text{(III.5-9)}$$

The computer print-out for a starting gain of zero is given in Figure III.5-6 and is compared with an I.C. in Figure III.5-7. There is an offset in Figure III.5-7 between the average costs produced by ISAFT and those produced by the I.C. This is due to the difficulty in approximating a uniform distribution on the surface of a sphere by a finite number of discrete points. (This difficulty increases rapidly with the order of the system). It is nevertheless clear that ISAFT achieved a minimum average cost.

III.6 Comments on the Finite Time Problem

The objective of the I.S.A. approach is to determine the feedback gains giving the minimum value of the gain function GF, which is the expected value of the cost functional. The I.S.A. algorithm presented in Section III.3, however, is designed only to determine feedback gains

```
 STATES     CONTROLS     FEEDBACKS     IGAINS
      3            1             1          0

SYSTEM MATRIX A
      0.0000000      1.0000000      0.0000000      0.0000000
      0.0000000      1.0000000      6.0000000      7.0000000
      0.0000000

CONTROL MATRIX B
      0.0000000      0.0000000      1.0000000

TERMINAL TIME T =        1.0000000

TERMINAL COST MATRIX F
      3.0000000      0.0000000      0.0000000      0.0000000
      2.0000000      0.0000000      0.0000000      0.0000000
      1.0000000

STATE WEIGHTING MATRIX Q
      1.0000000      0.0000000      0.0000000      0.0000000
      2.0000000      0.0000000      0.0000000      0.0000000
      3.0000000

CONTROL WEIGHTING MATRIX R
      0.1000000
```

FIGURE III.5-6 Computer Print-Out For Example 3

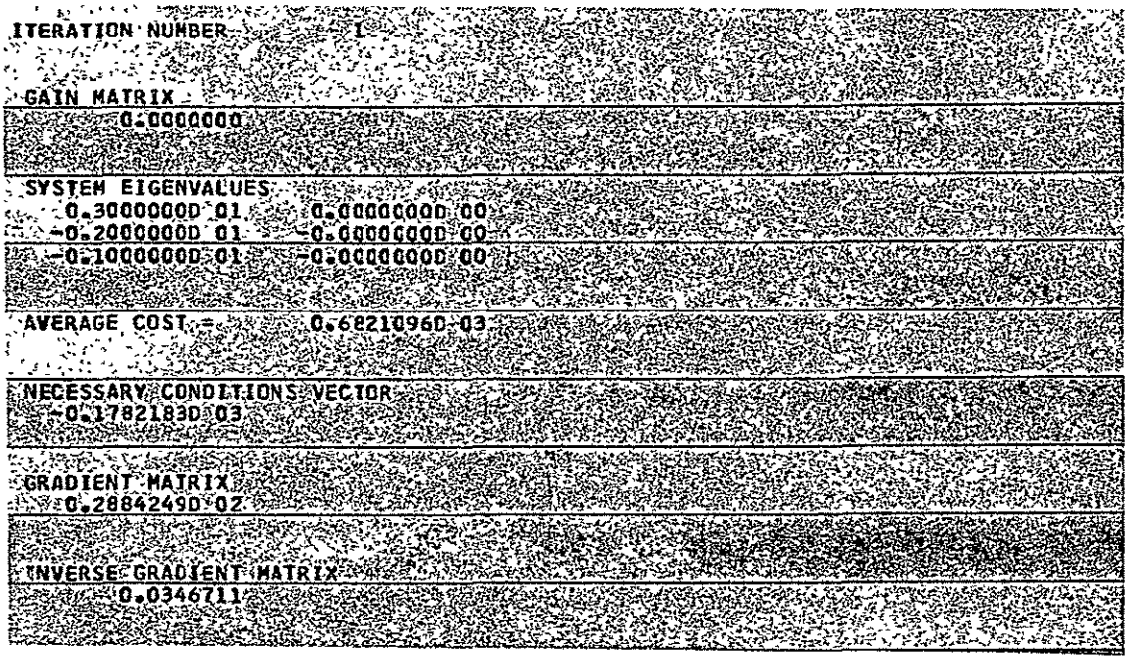


FIGURE III.5-6 (Cont'd)

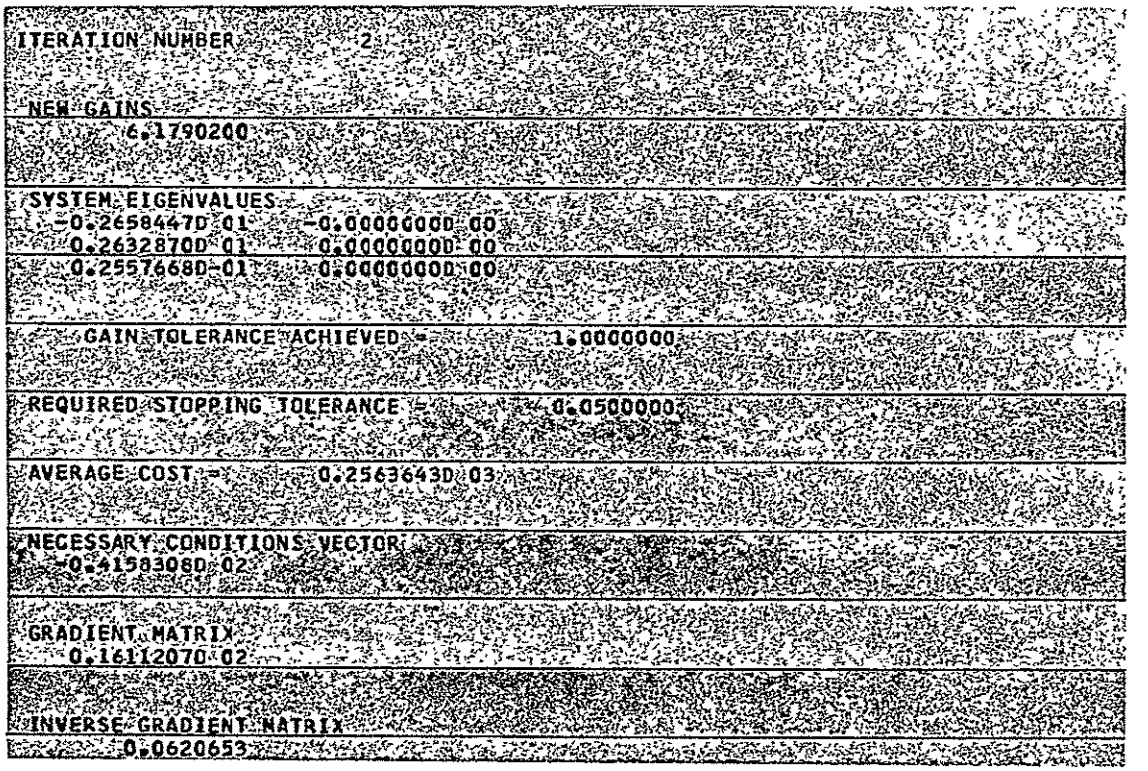


FIGURE III.5-6 (Cont'd)

```
ITERATION NUMBER          3

NEW GAINS
          8.7598845

SYSTEM EIGENVALUES
 -0.2824387D 01    -0.0000000D 00
  0.2420721D 01     0.0000000D 00
  0.4036657D 00    -0.0000000D 00

     GAIN TOLERANCE ACHIEVED =          0.2946231

REQUIRED STOPPING TOLERANCE =          0.0500000

  AVERAGE COST =        0.2174369D 03

NECESSARY CONDITIONS VECTOR
 -0.5427128D 01

GRADIENT MATRIX
  0.1201729D 02

INVERSE GRADIENT MATRIX
          0.0832135
```

FIGURE III.5-6 (Cont'd)

```
ITERATION NUMBER        4

NEW GAINS
     9.2114946

SYSTEM EIGENVALUES
  -0.28507120 01     -0.00000000 00
   0.23767130 01      0.00000000 00
   0.47399860 00      0.00000000 00

 GAIN TOLERANCE ACHIEVED =          0.0490268

REQUIRED STOPPING TOLERANCE =       0.0500000

SOLUTION IS COMPLETE. ABOVE GAINS ARE OPTIMAL

AVERAGE COST =       0.21660510 03

COMPILE TIME=    26.74 SEC,EXECUTION TIME=   976.35 SEC,OBJECT CODE=   64216 BYTES
```


FIGURE III.5-6 (Cont'd.)

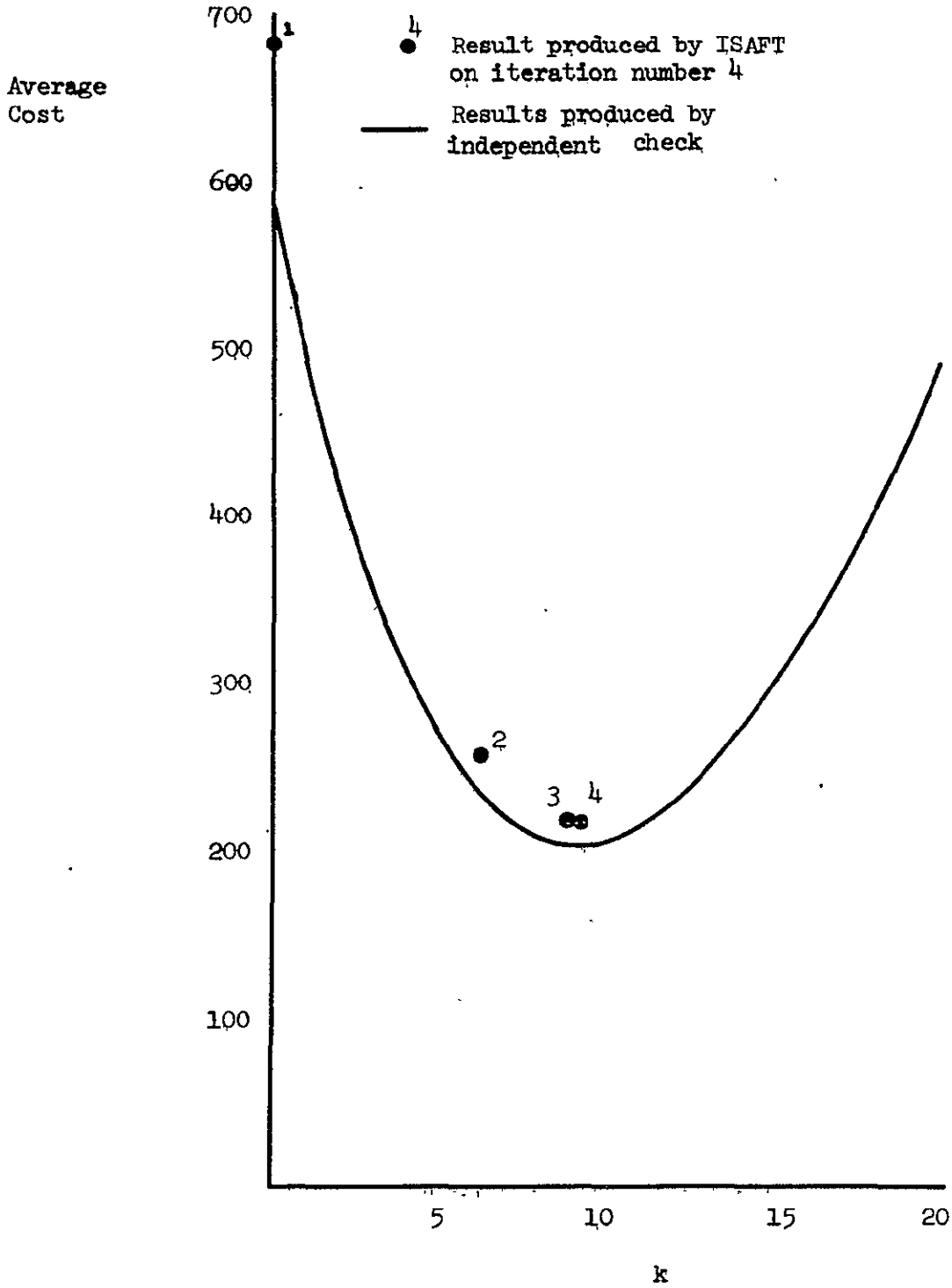


FIGURE III.5-7 Verification of the ISAFT results for Example 3

satisfying certain necessary conditions. The necessary conditions are that the partial derivatives of the gain function with respect to each feedback gain should be zero. These necessary conditions may be satisfied by local maxima, minima or points of inflection. With a multidimensional problem points of inflection seldom occur in practice. The convergence factor μ built into the digital computer program ISAFT prevents convergence to a local maximum. There still remains the problem of convergence to a local rather than a global minimum. No solution to this problem was found. It is felt, however, that it should be feasible to guarantee the existence of a single (global) minimum for suitable system and cost matrices, and useful work could be done in this area.

The digital computer program ISAFT computes the various required functions of the state transition matrix (see (III.3-33)) through (III.3-38)) analytically. It was felt that this best illustrated the procedure. For high order systems however, it should be more economical to evaluate the state transition matrix at a number of discrete time instants and then evaluate the required functions by numerical integration.

CHAPTER IV

SUMMARY AND CONCLUSIONS

This report has considered the determination of optimal time-invariant output-feedback controllers for linear dynamic systems with quadratic cost functionals. The need for such controllers in practical engineering was illustrated, and the contributions of earlier researchers in the field were reviewed.

It was determined that deficiencies existed in two areas. For the case where optimization was to take place over the semi-infinite time interval Levine[19] and Cassidy[20] had each derived a suitable computational algorithm. Both algorithms, however, required initialization by suitably stabilizing feedback gains and neither author gave a method for determination of such gains. For the case of optimization over a finite time interval, no satisfactory existing techniques were uncovered.

The problem of the determination of stabilizing feedback gains was approached via a gradient technique. The technique evolved from an eigenvalue sensitivity relationship given in Fadeev and Fadeeva[27]. It was mechanized in the digital computer program GRADGN (Appendix I) and provides a practical method for determination of local stability maxima in feedback gain space. It is felt that this gradient technique should complement the existing optimization algorithms for the semi-infinite time interval problem.

A new technique was derived for the finite time interval problem. The technique is based on the Initial State Averaging concept, previously used for somewhat different problems by Levine[19] and by Kleinman, Fortmann, and Athans[25]. A computational algorithm was derived and is incorporated in the digital computer program ISAFT (Appendix II). The algorithm satisfies a

set of necessary conditions by Newton-Raphson iteration, using their gradient with respect to the feedback gains. This is equivalent to minimization of the expected value of the cost functional by the method of second variations.

The contributions made by this report should aid in the search for practical optimal controllers.

Several outstanding problems remain. Foremost among these is the question of the sufficiency of the solutions obtained by the optimization technique described above. The technique was designed to determine local minima of the expected value of the cost functional. An examination of the convexity of this quantity in feedback gain space might uncover conditions ensuring a single (global) minimum. The gain initialization technique is similarly local and would also benefit by extension to a global technique.

The computational methods used in the digital computer program ISAFT were designed to illustrate the theory. They are satisfactory for low order systems, but not for high order systems. It is felt that relatively simple modifications to ISAFT should eliminate this deficiency.

Bibliography

1. DeRusso, P.M., R.J. Roy, and C.M. Close, "State Variables for Engineers," John Wiley and Sons, New York, 1965.

2. Athans, M. and P.L. Falb, "Optimal Control", McGraw-Hill Book Company, New York, 1966.

3. Schultz, D.G. and J.L. Melsa, "State Functions and Linear Control Systems", McGraw-Hill Book Company, New York, 1967.

4. Sage, A.P., "Optimum Systems Control", Prentice-Hall, Inc., Englewood Cliffs, N.J., 1968.

5. D'Azzo, J.J. and C.H. Houpis, "Feedback Control System Analysis and Synthesis", McGraw-Hill Book Company, New York, 1960.

6. Nyquist, H., "Regeneration Theory", Bell System Tech. J., 1932.

7. Bode, H.W., "Network Analysis and Feedback Amplifier Design", D. Van Nostrand Company, Inc., Princeton, N.J., 1945.

8. Evans, W.R., "Graphical Analysis of Control Systems", Trans. AIEE, Vol. 67, 1948.

9. Wiener, N., "Extrapolation, Interpolation, and Smoothing of Stationary Time Series", John Wiley and Sons, New York, 1949.

10. Newton, G.C. Jr., L.A. Gould and J.F. Kaiser, "Analytical Design of Linear Feedback Controls", John Wiley and Sons, New York, 1957.

11. Pontryagin, L.S., V. Boltyanskii, R. Gamkrelidze, and E. Mishchenko, "The Mathematical Theory of Optimal Processes", Interscience Publishers Inc., New York, 1962.

12. Kalman, R.E., "Contributions to the Theory of Optimal Control", Bol. Soc. Mat. Mex., Vol. 5, pp.102-119, 1960.

13. Kalman, R.E., and R.S. Bucy, "New Results in Linear Filtering and Prediction Theory", J. Basic Eng., ser. D, Vol. 83, pp.95-108, March, 1961.

14. Luenberger, D.G., "Determining the State of a Linear System with Observers of Low Dynamic Order", Ph.D. dissertation, Dept. of Elec. Eng., Stanford Univ., Calif. 1963.

15. Ash. R.H., Jr., "State Estimation in Linear Systems - A Unified Theory of Minimum Order Observers", Ph.D. dissertation, Systems Engineering Division, Rensselaer Polytechnic Institute, Troy, New York, 1969.

16. Hass, V. and S. Martuza, "On the Design of Specific Optimal Controllers", Purdue University, School of Electrical Engineering, May, 1967.

17. Agarwal, G., and R. Sridhar, "Design of Specific Optimal Control Systems", IEEE Region Six Conference, Tucson, Arizona, 1966.

18. Rekasius, Z., "Optimal Linear Regulators with Incomplete State Feedback", IEEE Trans. on Automatic Control, Vol. AC-12 pp. 296-299, June, 1967.

19. Levine, W.S., "Optimal Output-Feedback Controllers for Linear Systems", Ph.D. dissertation, Dept. of Electrical Engineering, Massachusetts Institute of Technology, Cambridge, Mass., 1969.

20. Cassidy, J.F., Jr., "Optimal Control with Unavailable States", Ph.D dissertation, Systems Engineering Division, Rensselaer Polytechnic Institute, Troy, New York, 1969.

21. Koenigsberg, W.D., "Output Feedback Control with Application to Unstable Linear Systems." Ph.D. dissertation, Rensselaer Polytechnic Institute, Troy, New York, 1969.

22. Jameson, A., "Design of a Single Input System for Specified Roots Using Output Feedback", IEEE Trans. On Automatic Control, Vol. AC-15, pp. 345-348, June, 1970.

23. Davison, E.J., "On Pole Assignment in Linear Systems with Incomplete State Feedback", IEEE Trans. on Automatic Control, Vol. AC-15, pp. 348-351, June 1970.

24. Kleinman, D.L., and M. Athans, "The Design of Suboptimal Linear Time Varying Systems", IEEE Trans. on Automatic Control, Vol. AC-13, pp. 150-159, April, 1968.

25. Kleinman, D.L., T.Fortmann, and M. Athans, "On the Design of Linear Systems with Piecewise Constant Feedback Gains", IEEE Trans. on Automatic Control, Vol. AC-13, pp. 354-362, August, 1968.

26. McBrinn, D.E. and L. Willner, "Demonstration of Design Techniques", Final Report, Contract Number NAS8-21131, Rensselaer Polytechnic Institute, Troy, New York, November, 1969.

27. Fadeev, D.K. and V.N. Fadeeva, "Computational Methods of Linear Algebra", Freeman, San Francisco, California, 1963, pp.228-229.

28. "IBM System/360 Scientific Subroutine Package (360A-CM-03X), Version III. Application Description", I.B.M., White Plains, N.Y., 1968.

29. Van Ness, J.E., "Inverse Iteration Method for Finding Eigenvectors", IEEE Trans. on Automatic Control, Vol. AC-14, pp. 63-66, February, 1969.

30. Reddy, D.C., "Sensitivity of an Eigenvalue of a Multivariable Control System", Electronics Letters, pp. 446, Vol. 2, No. 12, December, 1966.

31. Levine, W.S., and M. Athans, "On the Determination of the Optimal Constant Output Feedback Gains for Linear Multivariable Systems", IEEE Transactions on Automatic Control, Vol. AC-15, pp. 44-48, February, 1970.

APPENDIX I

This Appendix comprises a listing of the gain initialization digital computer program GRADGN.

```
    /JOB          4257      MCBRINN,LINES=55
 1        DIMENSION W(30,4),XR(30),XI(30),VR(30),VI(30),IROW(30,2),
          1  GRADR(30,30),GRADI(30,30),GRAD(30,30),VR.N(30),VIN(30),
          2  G(30,30)
 2        COMMON AMAT(30,30),BMAT(30,30),AHAT(30,30),AAAA(900),ASQR(30,30),
          1  RR(30),RI(30),IANA(30)
 3        REAL K(30,30)
 4   10   FORMAT (7I10)
 5   20   FORMAT (4E13.6)
 6   25   FORMAT (20A4)
 7   30   FORMAT (7F10.4)
 8   35   FORMAT ('1',T4,'STATES',4X,'CONTROLS',4X,'FEEDBACKS',4X,'IGAINS',
          1 4X,'IDELR',4X,'ISTOP')
 9   36   FORMAT (//T3,'STABILITY INCREASE INCREMENT =',F20.8)
10   37   FORMAT (//T3,'MINIMUM STABILITY REQUIRED FOR TERMINATION',3X,F20.8
          1)
11   40   FORMAT (//T3,'SYSTEM MATRIX AMAT')
12   45   FORMAT(//T3,'CONTROL MATRIX BMAT')
13   50   FORMAT (//T2,'MAXIMUM REAL PART OF ROOTS')
14   60   FORMAT (//T3,'EIGVEC ERROR MESSAGES')
15   65   FORMAT (T3,'Sml=',E10.4,10X,'ITER=',I5,10X,'DIF=',E10.4)
16   70   FORMAT (//T3,'EIGENVECTORS CORRES TO MRP EIGENVALUE')
17   75   FORMAT (T6,'ROW REAL PART',5X,'ROW IMAG PART',5X,'COL REAL PART',
          1 5X,'COL IMAG PART')
18   80   FORMAT (//T3,'GRADIENT MATRIX')
19   82   FORMAT ('1',T3,'PROGRAM TERMINATION DUE TO EXCESSIVE NUMBER OF ITE
          1RATIONS')
20   84   FORMAT ('1',T3,'SPECIFIED STABILITY HAS BEEN ATTAINED')
21   86   FORMAT ('1',T3,'GAIN SEARCH PROCEEDS ALONG A NEW GRADIENT. ITERATI
          1ON NUMBER',3X,I3)
22   87   FORMAT (//T3,'GAINS TOO FAR FROM STARTING POINT. COMPUTE NEW GRADI
          1ENT')
23   88   FORMAT (//T3,'STABILITY INCREASE STEP SIZE =',F20.8)
24   90   FORMAT (//T3,'STEP SIZE IS DOUBLED')
25   92   FORMAT (//T3,'CONVEX CURVE FIT TO STABILITY LOCUS')
26   93   FORMAT (//T3,'STEP SIZE TOO LARGE. RETURN TO PREVIOUS BEST GAINS')
27   94   FORMAT (//T3,'CURVE FIT NOT SUCCESSFUL. RETURN TO PREVIOUS BEST GA
          1INS')
28   95   FORMAT (//T3,'STEP SIZE TOO LARGE. COMPUTE NEW GRADIENT AT PREVIOU
          1S BEST GAINS')
29   96   FORMAT ('1',T3,'STABILITY IMPROVEMENT RATE TOO SLOW. PROGRAM TERMI
          1NATED.')
30   97   FORMAT (//T3,'LAST RESULTS SHOW MOST STABLE CONDITION FOUND')
31   98   FORMAT (//T3,'STEP SIZE TOO LARGE. PROGRAM CONTINUES WITH REDUCED
          1STEP SIZE')
     C
     C          NS = NUMBER OF SYSTEM STATES
     C          NC = NUMBER OF CONTROLS
     C          NF = NUMBER OF FEEDBACK STATES
     C      IGAINS = 0 - INITIAL FEEDBACK GAINS ARE ZERO
     C             = 1 - INITIAL FEEDBACK GAINS ARE READ IN
     C       IDELR = 0 - STABILITY INCREMENT = 0.1
     C             = 1 - STABILITY INCREMENT IS READ IN AS DELR
     C      ISTOP = 0 - PROGRAM MAXIMIZES SYSTEM STABILITY
     C             = 1 - PROGRAM TERMINATES AT STABILITY SPECIFIED BY STOPR
```

```
      C
   32       READ (1,10) NS,NC,NF,IGAINS,IDELR,ISTOP
   33       WRITE (3,35)
   34       WRITE (3,10) NS,NC,NF,IGAINS,IDELR,ISTOP
      C
      C      DELR CONTROLS STABILITY INCREASE INCREMENT
      C
   35       IF (IDELR) 120,120,110
   36  110  READ (1,30) DELR
   37       GO TO 130
   38  120  DELR=0.1
   39  130  DELRI=DELR
   40       WRITE (3,36) DELR
      C
      C      STOPR CONTROLS STABILITY REQUIRED FOR PROGRAM TERMINATION
      C
   41       IF (ISTOP) 150,150,140
   42  140  READ (1,30) STOPR
   43       GO TO 160
   44  150  STOPR=-1.0E6
   45  160  WRITE (3,37) STOPR
   46       DO 170 I=1,NS
   47       DO 170 J=1,NC
   48  170  K(I,J)=0.
      C
      C      SYSTEM DATA MATRICES ARE READ IN BY ROWS
      C       AMAT IS SYSTEM MATRIX
      C       BMAT IS CONTROL MATRIX
      C
   49       READ (1,30) ((AMAT(I,J),J=1,NS),I=1,NS)
   50       WRITE (3,40)
   51       WRITE (3,20) ((AMAT(I,J),J=1,NS),I=1,NS)
   52       READ (1,30) ((BMAT(I,J),J=1,NC),I=1,NS)
   53       WRITE (3,45)
   54       WRITE (3,20) ((BMAT(I,J),J=1,NC),I=1,NS)
      C
      C      INITIALIZATION OF GAIN MATRIX K
      C
   55       IF (IGAINS) 190,190,180
   56  180  READ (1,30) ((K(I,J),J=1,NC),I=1,NF)
   57  190  KOUNT=1
   58       ROOTT=0.
      C
      C       SUBROUTINE STAB COMPUTES THE SYSTEM STABILITY
      C
   59  195  CALL STAB(NS,NC,NF,K)
   60  198  IF(RR(1)-STOPR) 999,200,200
   61  200  KOUNT=KOUNT+1
   62       IF (KOUNT-10) 210,210,1100
      C
      C      GAIN SEARCH PROCEEDS ALONG A NEW GRADIENT
      C
   63  210  ROOTR=RR(1)
   64       ROOTI=RI(1)
   65       WRITE (3,50)
```

```
      WRITE (3,2)) ROOTR
      RTEST=ROOTT
      ROOTT=ROOTR
      K1=KOUNT-1
      WRITE (3,86) K1
C
C     COMPUTATION OF EIGENVECTORS
C
      CALL MSQ(NS)
      CALL EIGVEC(3,AHAT,ASQR,W,IROW,XR,XI,VR,VI,ROOTR,ROOTI,NS,30,0,
     1SW1,ITER,DIF,2)
C
C     SW1 = 0 FOR AN EXACT EIGENVALUE AND NO ROUND-OFF ERROR
C     ITER = NUMBER OF ITERATIONS USED TO FIND EIGENVECTORS.
C     IF TOLERANCE IS NOT ACHIEVED, PROGRAM ACCEPTS VALUES AT ITER = 15.
C     DIF = LARGEST CHANGE IN ANY EIGENVECTOR COMPONENT AT FINAL ITER.
C
      WRITE (3,60)
      WRITE (3,65) SW1,ITER,DIF
      WRITE (3,70)
      WRITE (3,75)
      WRITE (3,20) (VR(I),VI(I),XR(I),XI(I),I=1,NS)
C
C     NORMALISE EIGENVECTORS INNER PRODUCT
C
      VECMGR=0.
      VECMGI=0.
      DO 240 I=1,NS
      VECMGR=VECMGR+VR(I)*XR(I)-VI(I)*XI(I)
  240 VECMGI=VECMGI+VR(I)*XI(I)+VI(I)*XR(I)
      VECMGS=VECMGR*VECMGR+VECMGI*VECMGI
      DO 250 I=1,NS
      VRN(I)=(VR(I)*VECMGR+VI(I)*VECMGI)/VECMGS
  250 VIN(I)=(VI(I)*VECMGR-VR(I)*VECMGI)/VECMGS
C
C     COMPUTE GRADIENT MATRIX
C
      DO 300 J=1,NF
      DO 300 L=1,NC
      GRADR(J,L)=0.
      GRADI(J,L)=0.
      DO 290 I=1,NS
      GRADR(J,L)=GRADR(J,L)-VRN(I)*BMAT(I,L)
  290 GRADI(J,L)=GRADI(J,L)+VIN(I)*BMAT(I,L)
  300 GRAD(J,L)=GRADR(J,L)*XR(J)+GRADI(J,L)*XI(J)
      WRITE (3,80)
      WRITE (3,20) ((GRAD(I,J),J=1,NC),I=1,NF)
      WRITE (3,88) DELR
      GRDSQ=0.
      DO 320 I=1,NF
      DO 320 J=1,NC
  320 GRDSQ=GRDSQ+GRAD(I,J)*GRAD(I,J)
      DELK1=-DELR/GRDSQ
      DO 340 I=1,NF
      DO 340 J=1,NC
```

```
      G(I,J)=K(I,J)
  340 K(I,J)=G(I,J)+DELK1*GRAD(I,J)
      ROOT1=ROOT3
      CALL STAB(NS,NC,NF,K)
      IF (RR(1)-STOPR) 999,390,390
  390 ROOT2=RR(1)
      KOUNT1=1
      KOUNT2=1
      IF (ROOT2-ROOT1) 400,400,600
  400 IF((ROOT1-ROOT2)/DELR-.95) 420,420,500
  420 DELK2=.5*DELR*DELK1/(ROOT2-ROOT1+DELR)
      WRITE (3,92)
      DO 440 I=1,NF
      DO 440 J=1,NC
  440 K(I,J)=G(I,J)+DELK2*GRAD(I,J)
      CALL STAB(NS,NC,NF,K)
      IF(RR(1)-STOPR) 999,999,460
  460 ROOT3=RR(1)
      IF(ROOT3-ROOT2) 470,471,480
  470 IF(KOUNT-2) 475,475,472
  472 IF(RTEST-ROOT3-DELR1) 1200,475,475
  475 DELR=.5*(ROOT1-ROOT3)
      GO TO 200
  480 IF(KOUNT-2) 485,485,482
  482 IF(RTEST-ROOT2-DELR1) 483,485,485
  483 DO 484 I=1,NF
      DO 484 J=1,NC
  484 K(I,J)=G(I,J)+DELK1*GRAD(I,J)
      WRITE (3,93)
      CALL STAB(NS,NC,NF,K)
      GO TO 1200
  485 DELR=.5*(ROOT1-ROOT2)
      WRITE (3,94)
      DO 490 I=1,NF
      DO 490 J=1,NC
  490 K(I,J)=G(I,J)+DELK1*GRAD(I,J)
      GO TO 195
  500 DELK2=2.*DELK1
      KOUNT1=KOUNT1+1
      IF(KOUNT1-11) 505,505,502
  502 DELR=.5*(ROOT1-ROOT2)
      WRITE (3,87)
      GO TO 200
  505 WRITE (3,90)
      DO 510 I=1,NF
      DO 510 J=1,NC
  510 K(I,J)=G(I,J)+DELK2*GRAD(I,J)
      CALL STAB(NS,NC,NF,K)
      IF(RR(1)-STOPR) 999,999,520
  520 ROOT3=RR(1)
      IF (ROOT3-ROOT2) 530,530,580
  530 ROOT2=ROOT3
      DELK1=DELK2
      GO TO 400
  580 IF(KOUNT-2) 585,585,582
```

```
160     582  IF(RTEST-ROOT2-DELR1) 583,585,585
161     583  DO 584 I=1,NF
162          DO 584 J=1,NC
163     584  K(I,J)=G(I,J)+DELK1*GRAD(I,J)
164          WRITE (3,93)
165          CALL STAB(NS,NC,NF,K)
166          GO TO 1200
167     585. DELR=.5*(ROOT1-ROOT2)
168          WRITE (3,95)
169          DO 590 I=1,NF
170          DO 590 J=1,NC
171     590  K(I,J)=G(I,J)+DELK1*GRAD(I,J)
172          GO TO 195
173     600  KOUNT2=KOUNT2+1
174          IF(KOUNT2-6) 605,605,1100
175     605  DELK2=.5*DELR*DELK1/(ROOT2-ROOT1+DELR)
176          WRITE (3,92)
177          DO 620 I=1,NF
178          DO 620 J=1,NC
179     620  K(I,J)=G(I,J)+DELK2*GRAD(I,J)
180          CALL STAB(NS,NC,NF,K)
181          IF(RR(1)-STOPR) 999,999,640
182     640  ROOT3=RR(1)
183          IF(ROOT3-ROOT1) 680,680,660
184     660  DELK1=DELK2
185          WRITE (3,98)
186          ROOT2=ROOT3
187          GO TO 600
188     680  IF(KOUNT-2) 685,685,682
189     682  IF(RTEST-ROOT3-DELR1) 1200,685,685
190     685. DELR=.5*(ROOT1-ROOT3)
191          GO TO 200
192     999  WRITE (3,84)
193          GO TO 2000
194     1100 WRITE (3,82)
195          GO TO 2000
196     1200 WRITE (3,96)
197          WRITE (3,97)
198     2000 CONTINUE
199          STOP
200          END
```

```
      SUBROUTINE STAB(NS,NC,NF,K)
      COMMON AMAT(30,30),EMAT(30,30),AHAT(30,30),AAAA(900),ASOR(30,30),
     1       BB(30),RI(30),IANA(30)
      REAL K(30,30)
   10 FORMAT (//T3,'GAIN MATRIX K')
   20 FORMAT (4E18.7)
   30 FORMAT (//T2,'ROOTS',5X,'REAL PART',11X,'IMAG. PART')
   40 FORMAT (2E20.8)
      IA=NS
      WRITE (3,10)
      WRITE (3,20) ((K(I,J),J=1,NC),I=1,NF)
      CALL AMHT(NS,NC,K)
      CALL VECT(NS)
      CALL HSBG(NS,AAAA,IA)
      CALL ATEIG(NS,AAAA,BB,RI,IANA,IA)
      WRITE (3,30)
      WRITE (3,40) (BB(I),RI(I),I=1,NS)
      CALL MAXRT(NS)
      RETURN
      END
```

```
      SUBROUTINE AMAT(NS,NC,K)
C
C     COMPUTES AHAT = AMAT - BMAT*K(TRANSPOSE)
C
      COMMON AMAT(30,30),BMAT(30,30),AHAT(30,30),AAAA(900),ASQR(30,30),
     1       RR(30),RI(30),IANA(30)
      REAL K(30,30)
      DO 100 I=1,NS
      DO 100 J=1,NS
      AHAT(I,J)=AMAT(I,J)
      DO 100 L=1,NC
  100 AHAT(I,J)=AHAT(I,J)-BMAT(I,L)*K(J,L)
      RETURN
      END
```

```
      SUBROUTINE VECT(NS)
C
C     CONVERTS AHAT TO SINGLE SUBSCRIPT FORM AAAA
C
      COMMON AMAT(30,30),BMAT(30,30),AHAT(30,30),AAAA(900),ASQR(30,30),
     1       RR(30),RI(30),IANA(30)
      DO 100 J=1,NS
      DO 100 I=1,NS
      K=(J-1)*NS+I
  100 AAAA(K)=AHAT(I,J)
      RETURN
      END
```

```
      SUBROUTINE MSQ(NS)
C
C     COMPUTES ASQR = AHAT*AHAT
C
      COMMON AMAT(30,30),BMAT(30,30),AHAT(30,30),AAAA(900),ASQR(30,30),
     1       RR(30),RI(30),IANA(30)
      DO 100 I=1,NS
      DO 100 J=1,NS
      ASQR(I,J)=0.
      DO 100 K=1,NS
  100 ASQR(I,J)=ASQR(I,J)+AHAT(I,K)*AHAT(K,J)
      RETURN
      END
```

```
      SUBROUTINE HSBG(N,A,IA)
C
C     CONVERTS A TO UPPER HESSENBERG FORM
C
      DOUBLE PRECISION DABS,DFLOAT,DSIGN,DBLE,DEXP,DLOG,DLOG10,DATAN
     1,DSIN,DCOS,DSQRT,DTANH,DMOD,DMAX1,DMIN1
      DOUBLE PRECISION S
      DIMENSION A(9,9)
      L=N
      NIA=L*IA
      LIA=NIA-IA
   20 IF(L-3) 360,40,40
   40 LIA=LIA-IA
      L1=L-1
      L2=L1-1
      ISUB=LIA+L
      IPIV=ISUB-IA
      PIV=ABS(A(IPIV))
      IF(L-3) 90,90,50
   50 M=IPIV-IA
      DO 80 I=L,M,IA
      T=ABS(A(I))
      IF(T-PIV) 80,80,60
   60 IPIV=I
      PIV=T
   80 CONTINUE
   90 IF(PIV) 100,320,100
  100 IF(PIV-ABS(A(ISUB))) 180,180,120
  120 M=IPIV-L
      DO 140 I=1,L
      J=M+I
      T=A(J)
      K=LIA+I
      A(J)=A(K)
  140 A(K)=T
      M=L2-M/IA
      DO 160 I=L1,NIA,IA
      T=A(I)
      J=I-M
      A(I)=A(J)
  160 A(J)=T
  180 DO 200 I=L,LIA,IA
  200 A(I)=A(I)/A(ISUB)
      J=-IA
      DO 240 I=1,L2
      J=J+IA
      LJ=L+J
      DO 220 K=1,L1
      KJ=K+J
      KL=K+LIA
  220 A(KJ)=A(KJ)-A(LJ)*A(KL)
  240 CONTINUE
      K=-IA
      DO 300 I=1,N
      K=K+IA
```

```
300           LK=K+L1
301           S=A(LK)
302           LJ=L-IA
303           DO 280 J=1,L2
304           JK=K+J
305           LJ=LJ+IA
306       280 S=S+A(LJ)*A(JK)*1.0D0
307       300 A(LK)=S
308           DO 310 I=L,LIA,IA
309       310 A(I)=0.0
310       320 L=L1
311           GO TO 20
312       360 RETURN
313           END
```

```
      SUBROUTINE ATEIG(M,A,RR,RI,IANA,IA)
C
C     COMPUTES ROOTS OF UPPER HESSENBERG MATRIX A
C
      DOUBLE PRECISION DABS,DFLOAT,DSIGN,DBLE,DEXP,DLOG,DLOG10,DATAN
     1,DSIN,DCOS,DSQRT,DTANH,DMOD,DMAX1,DMIN1
      DIMENSION A(900),RR(30),RI(30),PRR(2),PRI(2),IANA(30)
      INTEGER P,P1,Q
      B7=1.0E-8
      E6=1.0E-6
      E10=1.0E-10
      DELTA=0.5
      MAXIT=30
      N=M
   20 N1=N-1
      IN=N1*IA
      NN=IN+N
      IF(N1) 30,1300,30
   30 NP=N+1
      IT=0
      DO 40 I=1,2
      PRR(I)=0.0
   40 PRI(I)=0.0
      PAN=0.0
      PAN1=0.0
      R=0.0
      S=0.0
      N2=N1-1
      IN1=IN-IA
      NN1=IN1+N
      N1N=IN+N1
      N1N1=IN1+N1
   60 T=A(N1N1)-A(NN)
      U=T*T
      V=4.0*A(N1N)*A(NN1)
      IF(ABS(V)-U*E7) 100,100,65
   65 T=U+V
      IF(ABS(T)-AMAX1(U,ABS(V))*E6) 67,67,68
   67 T=0.0
   68 U=(A(N1N1)+A(NN))/2.0
      V=SQRT(ABS(T))/2.0
      IF(T)140,70,70
   70 IF(U) 80,75,75
   75 RR(N1)=U+V
      RR(N)=U-V
      GO TO 130
   80 RR(N1)=U-V
      RR(N)=U+V
      GO TO 130
  100 IF(T)120,110,110
  110 RR(N1)=A(N1N1)
      RR(N)=A(NN)
      GO TO 130
  120 RR(N1)=A(NN)
      RR(N)=A(N1N1)
```

```
      130 RI(N)=0.0
          RI(N1)=0.0
          GO TO 160
      140 RR(N1)=U
          RR(N)=U
          RI(N1)=V
          RI(N)=-V
      160 IF(N2)1280,1280,180
      180 N1N2=N1N1-IA
          RMOD=RR(N1)*RR(N1)+RI(N1)*RI(N1)
          EPS=E10*SQRT(RMOD)
          IF(ABS(A(N1N2))-EPS)1280,1280,240
      240 IF(ABS(A(NN1))-E10*ABS(A(NN))) 1300,1300,250
      250 IF(ABS(PAN1-A(N1N2))-ABS(A(N1N2))*E6) 1240,1240,260
      260 IF(ABS(PAN-A(NN1))-ABS(A(NN1))*E6)1240,1240,300
      300 IF(IT-MAXIT) 320,1240,1240
      320 J=1
          DO 360 I=1,2
          K=NP-I
          IF(ABS(RR(K)-PRR(I))+ABS(RI(K)-PRI(I))-DELTA*(ABS(RR(K))
         1    +ABS(RI(K)))) 340,360,360
      340 J=J+I
      360 CONTINUE
          GO TO (440,460,460,480),J
      440 R=0.0
          S=0.0
          GO TO 500
      460 J=N+2-J
          R=RR(J)*RR(J)
          S=RR(J)+RR(J)
          GO TO 500
      480 R=RR(N)*RR(N1)-RI(N)*RI(N1)
          S=RR(N)+RR(N1)
      500 PAN=A(NN1)
          PAN1=A(N1N2)
          DO 520 I=1,2
          K=NP-I
          PRR(I)=RR(K)
      520 PRI(I)=RI(K)
          P=N2
          IF(N-3)600,600,525
      525 IPI=N1N2
          DO 580 J=2,N2
          IPI=IPI-IA-1
          IF(ABS(A(IPI))-EPS) 600,600,530
      530 IPIP=IPI+IA
          IPIP2=IPIP+IA
          D=A(IPIP)*(A(IPIP)-S)+A(IPIP2)*A(IPIP+1)+R
          IF(D)540,560,540
      540 IF(ABS(A(IPI)*A(IPIP+1))*(ABS(A(IPIP)+A(IPIP2+1)-S)+ABS(A(IPIP2+2)
         1 )) -ABS(D)*EPS) 620,620,560
      560 P=N1-J
      580 CONTINUE
      600 Q=P
          GO TO 680
```

```
      620 P1=P-1
          Q=P1
          IF(P1-1)680,680,650
      650 DO 660 I=2,P1
          IPI=IPI-IA-1
          IF(ABS(A(IPI))-EPS)680,680,660
      660 Q=Q-1
      680 II=(P-1)*IA+P
          DO 1220 I=P,N1
          III=II-IA
          IIP=II+IA
          IF(I-P)720,700,720
      700 IPI=II+1
          IPIP=IIP+1
          G1=A(II)*(A(III)-S)+A(IIP)*A(IPI)+R
          G2=A(IPI)*(A(IPIP)+A(II)-S)
          G3=A(IPI)*A(IPIP+1)
          A(IPI+1)=0.0
          GO TO 780
      720 G1=A(III)
          G2=A(III+1)
          IF(I-N2)740,740,760
      740 G3=A(III+2)
          GO TO 780
      760 G3=0.0
      780 CAP=SQRT(G1*G1+G2*G2+G3*G3)
          IF(CAP)800,860,800
      800 IF(G1)820,840,840
      820 CAP=-CAP
      840 T=G1+CAP
          PSI1=G2/T
          PSI2=G3/T
          ALPHA=2.0/(1.0+PSI1*PSI1+PSI2*PSI2)
          GO TO 880
      860 ALPHA=2.0
          PSI1=0.0
          PSI2=0.0
      880 IF(I-Q)900,960,900
      900 IF(I-P)920,940,920
      920 A(III)=-CAP
          GO TO 960
      940 A(III)=-A(III)
      960 IJ=II
          DO 1040 J=I,N
          T=PSI1*A(IJ+1)
          IF(I-N1)980,1000,1000
      980 IP2J=IJ+2
          T=T+PSI2*A(IP2J)
     1000 ETA=ALPHA*(T+A(IJ))
          A(IJ)=A(IJ)-ETA
          A(IJ+1)=A(IJ+1)-PSI1*ETA
          IF(I-N1)1020,1040,1040
     1020 A(IP2J)=A(IP2J)-PSI2*ETA
     1040 IJ=IJ+IA
          IF(I-N1)1080,1060,1060
```

```
      1060 K=N
           GO TO 1100
      1080 K=I+2
      1100 IP=IIP-I
           DO 1180 J=0,K
           JIP=IP+J
           JI=JIP-IA
           T=PSI1*A(JIP)
           IF(I-N1)1120,1140,1140
      1120 JIP2=JIP+IA
           T=T+PSI2*A(JIP2)
      1140 ETA=ALPHA*(T+A(JI))
           A(JI)=A(JI)-ETA
           A(JIP)=A(JIP)-ETA*PSI1
           IF(I-N1)1160,1180,1180
      1160 A(JIP2)=A(JIP2)-ETA*PSI2
      1180 CONTINUE
           IF(I-N2)1200,1220,1220
      1200 JI=II+3
           JIP=JI+IA
           JIP2=JIP+IA
           ETA=ALPHA*PSI2*A(JIP2)
           A(JI)=-ETA
           A(JIP)=-ETA*PSI1
           A(JIP2)=A(JIP2)-ETA*PSI2
      1220 II=IIP+1
           IT=IT+1
           GO TO 60
      1240 IF(ABS(A(NN1))-ABS(A(N1N2))) 1300,1280,1280
      1280 IANA(N)=0
           IANA(N1)=2
           N=N2
           IF(N2)1400,1400,20
      1300 RR(N)=A(NN)
           RI(N)=0.0
           IANA(N)=1
           IF(N1)1400,1400,1320
      1320 N=N1
           GO TO 20
      1400 RETURN
           END
```

```
      SUBROUTINE MAXRT(NS)
C
C     COMPUTES ROOT HAVING MAXIMUM REAL PART
C
      COMMON AMAT(30,30),BMAT(30,30),AHAT(30,30),AAAA(900),ASOR(30,30),
     1       RR(30),RI(30),IANA(30)
      DO 100 I=2,NS
      IF (RR(1)-RR(I))50,100,100
   50 RI(1)=RI(I)
      RR(1)=RR(I)
  100 CONTINUE
      RETURN
      END
```

```
      SUBROUTINE EIGVEC(IVC,  A, B, N, IROW, XR, XI, VR, VI, ROOTRE,    ESV1
     1   ROOTIE, NE, NMAX, T2, SW1, COUNTE, ERR,MMM)                     ESV1
C       SUBROUTINE TO FIND THE EIGENVECTORS OF A NON-SYMMETRIC MATRIX   ESV1
C       BY A MODIFIED WILKINSON'S INVERSE ITERATION METHOD.             ESV1
C       CONTROL IVC CODE IS                                             ESV1
C            1  FIND ONLY THE REGULAR EIGENVECTORS  (A X = LAMBDA X)    ESV1
C            2  FIND ONLY THE TRANSPOSED EIGENVECTORS (AT V = LAMBDA V) ESV1
C            3  FIND BOTH TYPES OF EIGENVECTORS.                        ESV1
      DIMENSION A(30,30),B(30,30),W(30,4),XR(30),XI(30),VR(30),VI(30),
     1IROW(30,2)
      INTEGER COUNT, COUNTE, T2                                         ESV1
      IO1=1
      IO3=3
      ROOTR = ROOTRE                                                    ESV1
      ROOTI = ROOTIE                                                    ESV1
      N = NE                                                            ESV1
      MM = MMM = 1                                                      ESV1
      N1 = N - 1                                                        ESV1
      NP1 = N + 1                                                       ESV1
      IVC1 = IVC - 1                                                    ESV1
      IVC2 = IVC1 - 1                                                   ESV1
      COUNT = 1                                                         ESV1
      DO 400 I=1,N                                                      ESV1
      W(I,1)=0.0E0
      XR(I)=0.0E0
  400 CONTINUE
      CLIM = 1.0E-4                                                     ESV1
      IF(ROOTI) 1, 60, 1                                                ESV1
C                                                                       ESV1
C     COMPLEX EIGENVALUE.                                               ESV1
C                                                                       ESV1
    1 TEMP = - ROOTR - ROOTR                                            ESV1
      ISW = 2                                                           ESV1
      TEMP2=ROOTR*ROOTR+ROOTI*ROOTI                                     ESV1
      JJ = 30                                                           ESV1
      DO 606 I = 1, N                                                   ESV1
      IF(T2) 600, 603, 600                                              ESV1
  600 DO 602 J = 1, N                                                   ESV1
      JJ = JJ + 1                                                       ESV1
      IF(JJ - 251) 602, 601, 601                                        ESV1
  601 JJ = 1                                                            ESV1
      READ (T2) (W(LL,1), LL = 1,250)                                   ESV1
  602 B(I,J) = A(I,J)*TEMP + W(JJ,1)                                    ESV1
      GO TO 605                                                         ESV1
  603 DO 604 J = 1, N                                                   ESV1
  604 B(I,J) = A(I,J)*TEMP + B(I,J)                                     ESV1
  605 B(I,I) = B(I,I) + TEMP2                                           ESV1
  606 A(I,I) = A(I,I) - ROOTR                                           ESV1
      IF(T2 .NE. 0) REWIND T2                                           ESV1
      GO TO 700                                                         ESV1
  607 IF(ICC) 622, 608, 622                                             ESV1
C                                                                       ESV1
C     MATRIX SINGULAR.                                                  ESV1
C                                                                       ESV1
  622 IF(IVC2) 623, 625, 623                                            ESV1
```

```
      623 DO 624 LL = 1, N                                                  ESY
          W(LL,2)=0.0
      624 XI(LL)=0.0
          IF(IVC1) 625, 514, 625                                            ESY
      625 DO 626 LL = 1, N                                                  ESY
          W(LL,4)=0.0
      626 VI(LL)=0.0
          GO TO 511                                                         ESY
      C                                                                     ESY
      C     MATRIX NOT SINGULAR.                                            ESY
      C                                                                     ESY
      608 DO 609 LL = 1, N                                                  ESY
          W(LL,1)=1.0
          W(LL,2)=1.0
          W(LL,3)=1.0
      609 W(LL,4)=1.0
      699 IF(IVC2) 610, 612, 610                                            ESY
      610 DO 611 I = 1, N                                                   ESY
          I2 = IROW(I,2)                                                    ESY
          XI(I2) = W(I,1)*ROOTI                                             ESY
          DO 611 J = 1, N                                                   ESY
      611 XI(I2) = XI(I2) + A(I,J)*W(J,2)                                   ESY
          IF(IVC1) 612, 500, 612                                            ESY
      612 DO 613 I = 1, N                                                   ESY
          VI(I) = W(I,3)*ROOTI                                              ESY
          DO 613 J = 1,N                                                    ESY
      613 VI(I) = VI(I) + A(J,I)*W(J,4)                                     ESY
          GO TO 499                                                         ESY
      615 CERR = 0.0                                                        ESY
          IF(IVC2) 616, 619, 616                                            ESY
      616 DO 618 I = 1, N                                                   ESY
          XR(I) = -W(I,2)                                                   ESY
          DO 617 J = 1, N                                                   ESY
      617 XR(I) = XR(I) + A(I,J)*XI(J)                                      ESY
      618 XR(I) = XR(I)/ROOTI                                               ESY
          IF(IVC1) 619, 633, 619                                            ESY
      619 DO 621 I = 1, N                                                   ESY
          VR(I) = -W(I,4)                                                   ESY
          DO 620 J = 1, N                                                   ESY
      620 VR(I) = VR(I) + A(J,I)*VI(J)                                      ESY
      621 VR(I) = VR(I)/ROOTI                                               ESY
      C                                                                     ESY
      C          SEARCH VECTORS FOR LARGEST ELEMENT AND NORMALIZE.          ESY
      C                                                                     ESY
      627 AMAX = 0.0                                                        ESY
          DO 629 L = 1, N                                                   ESY
          TEMP = VR(L)**2 + VI(L)**2                                        ESY
          IF(TEMP - AMAX) 629, 629, 628                                     ESY
      628 AMAX = TEMP                                                       ESY
          I2 = L                                                            ESY
      629 CONTINUE                                                          ESY
          C1 = VR(I2)/AMAX                                                  ESY
          C2 = -VI(I2)/AMAX                                                 ESY
          DO 630 L = 1, N                                                   ESY
          TEMP = VI(L)                                                      ESY
```

```
      VI(L) = VR(L)*C2 + TEMP*C1                                        ESY
  630 VR(L) = VR(L)*C1 - TEMP*C2                                        ESY
      IF(COUNT .EQ. 1) GO TO 632                                        ESY
      DO 631 LL = 1, N                                                  ESY
  631 CERR = AMAX1(CERR, ABS(VR(LL) - W(LL,3)), ABS(VI(LL) - W(LL,4)))  ESY
  632 IF(IVC2) 633, 638, 633                                            ESY
  633 AMAX = 0.0                                                        ESY
      DO 635 L = 1, N                                                   ESY
      TEMP = XR(L)**2 + XI(L)**2                                        ESY
      IF(TEMP - AMAX) 635, 635, 634                                     ESY
  634 AMAX = TEMP                                                       ESY
      I2 = L                                                            ESY
  635 CONTINUE                                                          ESY
      C1 = XR(I2)/AMAX                                                  ESY
      C2 = -XI(I2)/AMAX                                                 ESY
      DO 636 L = 1, N                                                   ESY
      TEMP = XI(L)                                                      ESY
      XI(L) = XR(L)*C2 + TEMP*C1                                        ESY
  636 XR(L) = XR(L)*C1 - TEMP*C2                                        ESY
      IF(COUNT .EQ. 1) GO TO 646                                        ESY
      DO 637 LL = 1, N                                                  ESY
  637 CERR = AMAX1(CERR, ABS(XR(LL) - W(LL,1)), ABS(XI(LL) - W(LL,2)))  ESY
C                                                                       ESY
C     TEST FOR CONVERGENCE.                                             ESY
C                                                                       ESY
  638 IF(COUNT .EQ. 1) GO TO 646                                        ESY
      IF(CERR .GE. 1.0E-4) GO TO 639                                    ESY
      IF(CERR .GE. CLIM) GO TO 648                                      ESY
      CLIM = CERR                                                       ESY
      IF(CLIM .LE. 1.0E-8) GO TO 648                                    ESY
  639 IF(COUNT .GE. 15) GO TO 68                                        ESY
  647 COUNT = COUNT + 1                                                 ESY
      IF(ROOTI) 642, 673, 642                                           ESY
  642 IF(IVC2) 640, 644, 640                                            ESY
  640 DO 641 LL = 1, N                                                  ESY
      W(LL,1) = XR(LL)                                                  ESY
  641 W(LL,2) = XI(LL)                                                  ESY
      IF(IVC1) 644, 610, 644                                            ESY
  644 DO 645 LL = 1, N                                                  ESY
      W(LL,3) = VR(LL)                                                  ESY
  645 W(LL,4) = VI(LL)                                                  ESY
      GO TO 699                                                         ESY
  646 CERR = 0.0                                                        ESY
      IF(ICC) 648, 647, 648                                             ESY
  648 ERR = CERR                                                        ESY
      COUNTE = COUNT                                                    ESY
      IF(ROOTI) 667, 668, 667                                           ESY
  667 DO 649 I = 1, N                                                   ESY
  649 A(I,I) = A(I,I) + ROOTR                                           ESY
      RETURN                                                            ESY
   68 PRINT 101, ROOTR, ROOTI, CERR                                     ESY
      GO TO 648                                                         ESY
C                                                                       ESY
C     REAL EIGENVECTORS.                                                ESY
C                                                                       ESY
```

```
      60 ISW = 1                                                        ESY
         DO 651 I = 1, N                                                ESY
         DO 650 J = 1, N                                                ESY
650   B(I,J) = A(I,J)                                                   ESY
651   B(I,I) = B(I,I) - ROOTR                                           ESY
      GO TO 700                                                         ESY
652   IF(ICC) 680, 685, 680                                             ESY
C                                                                       ESY
C     SINGULAR MATRIX.                                                  ESY
C                                                                       ESY
680   IF(IVC2) 681, 683, 681                                            ESY
681   DO 682 L = 1, N                                                   ESY
682   XI(L) = 0.0                                                       ESY
      IF(IVC1) 683, 514, 683                                            ESY
683   DO 684 L = 1, N                                                   ESY
684   VI(L) = 0.0                                                       ESY
      GO TO 511                                                         ESY
C                                                                       ESY
C     MATRIX NOT SINGULAR.                                              ESY
C                                                                       ESY
685   IF(IVC2) 653, 656, 653                                            ESY
653   DO 654 L = 1, N                                                   ESY
654   XI(L) = 1.0                                                       ESY
      IF(IVC1) 656, 500, 656                                            ESY
656   DO 657 L = 1, N                                                   ESY
657   VI(L) = 1.0                                                       ESY
      GO TO 499                                                         ESY
C                                                                       ESY
C     NORMALIZE REAL VECTORS.                                           ESY
C                                                                       ESY
655   CERR = 0.0                                                        ESY
      IF(IVC2) 658, 662, 658                                            ESY
658   C1=0.0                                                            ESY
      C2=0.0                                                            ESY
      DO 660 L = 1, N                                                   ESY
      TEMP = ABS(XI(L))                                                 ESY
      IF(TEMP - C1) 660, 660, 659                                       ESY
659   C1 = TEMP                                                         ESY
      C2 = XI(L)                                                        ESY
660   CONTINUE                                                          ESY
      DO 661 L = 1, N                                                   ESY
      XI(L) = XI(L)/C2                                                  ESY
      CERR = AMAX1(CERR, ABS(XI(L) - XR(L)))                            ESY
661   XR(L) = XI(L)                                                     ESY
      IF(IVC1) 662, 638, 662                                            ESY
662   C2=0.0                                                            ESY
      C1=0.0                                                            ESY
      DO 664 L = 1, N                                                   ESY
      TEMP = ABS(VI(L))                                                 ESY
      IF(TEMP - C1) 664, 664, 663                                       ESY
663   C1 = TEMP                                                         ESY
      C2 = VI(L)                                                        ESY
664   CONTINUE                                                          ESY
      DO 665 LL = 1, N                                                  ESY
      VI(LL) = VI(LL)/C2                                                ESY
```

```
      CERR = AMAX1(CERR, ABS(VI(LL) - W(LL,1)))                         ESY
      W(LL,1)=VI(LL)
  665 VR(LL)=W(LL,1)
      GO TO 638                                                         ESY
  668 IF(IVC2) 669, 671, 669                                            ESY
  669 DO 670 L = 1, N                                                   ESY
  670 XI(L) = 0.0                                                       ESY
      IF(IVC1) 671, 70, 671                                             ESY
  671 DO 672 L = 1, N                                                   ESY
  672 VI(L) = 0.0                                                       ESY
   70 RETURN                                                            ESY
  673 IF(IVC2) 674, 502, 674                                            ESY
  674 DO 675 I = 1, N                                                   ESY
      I2 = IROW(I,2)                                                    ESY
  675 XI(I2) = XR(I)                                                    ESY
      GO TO 500                                                         ESY
C                                                                       ESY
C        BACK SUBSTITUTION SECTION.                                     ESY
C                                                                       ESY
  499 IF(IVC2) 500, 502, 500                                            ESY
  500 DO 501 I = 2, N                                                   ESY
      I1 = I - 1                                                        ESY
      DO 501 J = 1, I1                                                  ESY
  501 XI(I) = XI(I) - B(I,J)*XI(J)                                      ESY
  511 IF(IVC1) 502, 514, 502                                            ESY
  502 DO 510 I = 1, N                                                   ESY
      I1 = I - 1                                                        ESY
      IF(I1) 503, 505, 503                                              ESY
  503 DO 504 J = 1, I1                                                  ESY
  504 VI(I) = VI(I) - B(I,J)*VI(J)                                      ESY
      IF(ICC) 505, 506, 505                                             ESY
  505 IF(B(I,I)) 506, 507, 506                                          ESY
  506 VI(I) = VI(I)/B(I,I)                                              ESY
      GO TO 510                                                         ESY
  507 IF(VI(I)) 508, 509, 508                                           ESY
  508 VI(I) = VI(I)*1.0E+15                                             ESY
      GO TO 510                                                         ESY
  509 VI(I) = 1.0                                                       ESY
  510 CONTINUE                                                          ESY
      IF(IVC2) 514, 525, 514                                            ESY
  514 DO 522 I = 1, N                                                   ESY
      IR = NP1 - I                                                      ESY
      IF(I - 1) 515, 517, 515                                           ESY
  515 I2 = IR + 1                                                       ESY
      DO 516 J = I2, N                                                  ESY
  516 XI(IR) = XI(IR) - B(IR,J)*XI(J)                                   ESY
      IF(ICC) 517, 518, 517                                             ESY
  517 IF(B(IR,IR)) 518, 519, 518                                        ESY
  518 XI(IR) = XI(IR)/B(IR,IR)                                          ESY
      GO TO 522                                                         ESY
  519 IF(XI(IR)) 520, 521, 520                                          ESY
  520 XI(IR) = XI(IR)*1.0E+15                                           ESY
      GO TO 522                                                         ESY
  521 XI(IR) = 1.0                                                      ESY
  522 CONTINUE                                                          ESY
```

```
      IF(INC1) 525, 529, 525                                            ESY
  525 DO 526 I = 2, N                                                   ESY
      IR = NP1 - I                                                      ESY
      I2 = IR + 1                                                       ESY
      DO 526 J = I2, N                                                  ESY
  526 VI(IR) = VI(IR) - B(J,IR)*VI(J)                                   ESY
      DO 527 L = 1, N                                                   ESY
      I2 = IROW(L,1)                                                    ESY
  527 VR(I2) = VI(L)                                                    ESY
      DO 528 L = 1, N                                                   ESY
  528 VI(L) = VR(L)                                                     ESY
  529 IF(ROOT1) 615, 655, 615                                           ESY
C                                                                       ESY
C     FACTOR MATRIX.                                                    ESY
C                                                                       ESY
  700 ICC = 0                                                           ESY
      SW1=1.0E72                                                        ESY
      DO 701 LL = 1, N                                                  ESY
  701 IROW(LL,1) = LL                                                   ESY
      DO 708 K = 1, N1                                                  ESY
      AMAX = ABS(B(K,K))                                                ESY
      IMAX = K                                                          ESY
      K1 = K + 1                                                        ESY
      DO 702 I = K1, N                                                  ESY
      IF(AMAX .GT. ABS(B(I,K))) GO TO 702                               ESY
      AMAX = ABS(B(I,K))                                                ESY
      IMAX = I                                                          ESY
  702 CONTINUE                                                          ESY
      IF(AMAX .LT. SW1) SW1 = AMAX                                      ESY
      IF(AMAX .GE. 1.0E-25) GO TO 723                                   ESY
      B(K,K) = 0.0                                                      ESY
      ICC = ICC + 1                                                     ESY
      GO TO 708                                                         ESY
  723 IF(IMAX .EQ. K) GO TO 704                                         ESY
      DO 703 J = 1, N                                                   ESY
      AMAX = B(K,J)                                                     ESY
      B(K,J) = B(IMAX,J)                                                ESY
  703 B(IMAX,J) = AMAX                                                  ESY
      I2 = IROW(K,1)                                                    ESY
      IROW(K,1) = IROW(IMAX,1)                                          ESY
      IROW(IMAX,1) = I2                                                 ESY
  704 DO 707 I = K1, N                                                  ESY
      IF(B(I,K)) 705, 707, 705                                          ESY
  705 B(I,K) = B(I,K)/B(K,K)                                            ESY
      DO 706 J = K1, N                                                  ESY
  706 B(I,J) = B(I,J) - B(K,J)*B(I,K)                                   ESY
  707 CONTINUE                                                          ESY
  708 CONTINUE                                                          ESY
      AMAX = ABS(B(N,N))                                                ESY
      IF(AMAX - 1.0E-25) 712, 712, 713                                  ESY
  712 B(N,N)=0.0                                                        ESY
      SW1=0.0                                                           ESY
      ICC = ICC + 1                                                     ESY
      GO TO 709                                                         ESY
  713 IF(AMAX .LT. SW1) SW1 = AMAX                                      ESY
```

```
      709 IF(ICC .LE. IOW) GO TO 710                                    ESY
          IF(MM) 1050,1050,1051
 1051 WRITE(103,1052) ICC
          COUNTE = 0                                                    ESY
          RETURN                                                        ESY
 1050 WRITE(103,1052) ICC
      710 DO 711 LL = 1, N                                              ESY
          I2 = IROW(LL,1)                                               ESY
      711 IROW(I2,2) = LL                                               ESY
          IF(ROOTI) 607, 652, 607                                       ESY
 1052 FORMAT(///23H ****** WARNING ******    ,'  SUBROUTINE EIGVEC HAS ESY
     1FOUND AN EIGENVALUE OF APPARENT MULTIPLICITY',                    ESY
     1                                   I4,/23X,'  COMPUTATION OF EIESY
     2GENVECTOR(S) CONTINUES AT USER S OPTION'//)                       ESY
 1051 FORMAT(38H )MORE THAN 15 LOOPS FOR EIGENVECTOR OF,2E12.4,         ESY
     2   14H DIFFERENCE OF,E12.4)                                       ESY
  102 FORMAT(16HO****WARNING**** , I4, 71H ZEROS ON DIAGONAL OF FACTOREDESY
     1 MATRIX.  CHECK FOR MULTIPLE EIGENVALUES./20X,                    ESY
     2' SUBROUTINE EIGVEC WILL NOT PERFORM COMPUTATION FOR THIS EIGENVECESY
     3TOR '//)                                                          ESY
       END
```

```
/DATA
```

APPENDIX II

This Appendix comprises a listing of the I.S.A. optimization digital computer program ISAFT.

```
/JOB          4257      MCPRINA,LINES=55
C
C         PROGRAM ISAFT
C         DETERMINES OPTIMAL CONSTANT GAIN OUTPUT FEEDBACK CONTROLLERS
C         FOR FINITE TIME STATE REGULATOR PROBLEMS
C         OUTPUTS ARE ASSUMED TO BE FIRST NF STATES
C
 1          COMMON M(3,3),MI(3,3),RC(3),AHAT(3,3),T,K(3,3),G(3,3),CHAT(3,3),
           1       FT(3,3),FFTI(3,3),DFT(3,3,3,3),FDFTI(3,3,3,3),
           2       D2FT(3,3,3,3,3,3),DFDFTI(3,3,3,3,3,3),FD2FTI(3,3,3,3,3,3),
           3       VW(9),VGRAD(9,9),GRADI(9,9),B(3,3),F(3,3),R(3,3),NS,NC,NF
 2          DIMENSION A(3,3),C(3,3)
 3          COMPLEX*16 M,MI,RC
 4          DOUBLE PRECISION AHAT,T,K,G,CHAT,FT,FFTI,DFT,FDFTI,D2FT,DFDFTI,
           1                 FD2FTI,VW,VGRAD,GRADI
 5          DOUBLE PRECISION GF,GF2,TEST,TEST1,GASTCP
 6          DOUBLE PRECISION DMAX1,DABS
 7    10    FORMAT (4F18.7)
 8    11    FORMAT (4D18.7)
 9    15    FORMAT (7I10)
10    20    FORMAT (14I5)
11    25    FORMAT ('1',T4,'STATES',4X,'CONTROLS',4X,'FEEDBACKS',4X,'IGAINS')
12    30    FORMAT (//T3,'INVERSE GRADIENT MATRIX')
13    35    FORMAT (//T3,'SYSTEM MATRIX A')
14    40    FORMAT (//T3,'NEW GAINS')
15    45    FORMAT (//T3,'CONTROL MATRIX B')
16    50    FORMAT (//T3,'GAIN MATRIX')
17    55    FORMAT (//T3,'TERMINAL TIME T =',F18.7)
18    60    FORMAT (//T7,'GAIN TOLERANCE ACHIEVED =',F18.7)
19    62    FORMAT (//T3,'AVERAGE COST =',D2 .7)
20    65    FORMAT (//T3,'TERMINAL COST MATRIX F')
21    70    FORMAT (//T3,'REQUIRED STOPPING TOLERANCE =',F18.7)
22    75    FORMAT (//T3,'STATE WEIGHTING MATRIX Q')
23    80    FORMAT (//T3,'CONTROL WEIGHTING MATRIX R')
24    85    FORMAT ('1','ITERATION NUMBER',I10)
25    90    FORMAT (//T3,'SOLUTION IS COMPLETE. ABOVE GAINS ARE OPTIMAL')
26    92    FORMAT (//T3,'STEP SIZE TOO LARGE. NEW AVERAGE COST WOULD HAVE BEE
           1N',D16.7)
27    93    FORMAT (T3,'GAIN ADJUSTMENT IS HALVED')
28    95    FORMAT (//T3,'CONVERGENCE TOO SLOW. PROGRAM TERMINATED')
29   100    READ (1,20) NS,NC,NF,IGAINS
30          READ (1,10) ((A(I,J),J=1,NS),I=1,NS)
31          READ (1,10) ((B(I,J),J=1,NC),I=1,NS)
32          READ (1,11) T
33          READ (1,10) ((F(I,J),J=1,NS),I=1,NS)
34          READ (1,10) ((Q(I,J),J=1,NS),I=1,NS)
35          READ (1,10) ((R(I,J),J=1,NC),I=1,NC)
36          READ (1,11) GASTCP
37          WRITE (3,25)
38          WRITE (3,15) NS,NC,NF,IGAINS
39          WRITE (3,35)
40          WRITE (3,10) ((A(I,J),J=1,NS),I=1,NS)
41          WRITE (3,45)
42          WRITE (3,10) ((B(I,J),J=1,NC),I=1,NS)
43          WRITE (3,55) T
```

```
      WRITE (3,65)
      WRITE (3,11) ((F(I,J),J=1,NS),I=1,NS)
      WRITE (3,75)
      WRITE (3,11) ((Q(I,J),J=1,NS),I=1,NS)
      WRITE (3,80)
      WRITE (3,11) ((R(I,J),J=1,NC),I=1,NC)
      NFP1=NF+1
      NFC=NF*NC
      DO 1000 I=1,NS
      DO 1000 J=1,NC
      G(I,J)=0.000
 1000 K(I,J)=0.000
      IF(IGAINS) 1200,1200,1100
 1100 READ (1,11) ((K(I,J),J=1,NC),I=1,NF)
 1200 CONTINUE
      DO 1220 I=1,NS
      DO 1220 J=NF,NS
      QHAT(I,J)=Q(I,J)
 1220 AHAT(I,J)=A(I,J)
      IT=1
      WRITE (3,85) IT
      WRITE (3,50)
      WRITE (3,11) ((K(I,J),J=1,NC),I=1,NF)
 1230 CONTINUE
      DO 1250 I=1,NS
      DO 1250 J=1,NF
      AHAT(I,J)=A(I,J)
      QHAT(I,J)=Q(I,J)
      DO 1250 N1=1,NC
      AHAT(I,J)=AHAT(I,J)-B(I,N1)*K(J,N1)
      DO 1250 N2=1,NC
 1250 QHAT(I,J)=QHAT(I,J)+K(I,N1)*R(N1,N2)*K(J,N2)
      CALL STRAP
 1260 CALL PEPA
      IT=IT+1
      CALL GAINPA(GP)
      WRITE (3,62) GP
      CALL NESCON
      CALL GRADNT
      CALL INVERT(VGRAD,GRADI,NFC)
      WRITE (3,30)
      WRITE (3,11) ((GRADI(I,J),J=1,NFC),I=1,NFC)
      CALL NEWRIT
      WRITE (3,85) IT
      WRITE (3,40)
      WRITE (3,11) ((G(I,J),J=1,NC),I=1,NF)
      IT1=0
 1280 CONTINUE
      DO 1300 I=1,NS
      DO 1300 J=1,NF
      AHAT(I,J)=A(I,J)
      QHAT(I,J)=Q(I,J)
      DO 1300 N1=1,NC
      AHAT(I,J)=AHAT(I,J)-B(I,N1)*G(J,N1)
      DO 1300 N2=1,NC
```

```
 99    1350 CHAT(I,J)=CHAT(I,J)+G(I,N1)*F(N1,N2)*G(J,N2)
100         CALL STRAM
101         CALL GAIN2(GF2)
102         IF(GF-GF2.GT.  .001) GO TO 135
103         WRITE (3,92) GF2
104         WRITE (3,93)
105         DO 1350 I=1,NF
106         DO 1350 J=1,NC
107    1350 G(I,J)=G(I,J)/2.+K(I,J)/2.
108         WRITE (3,42)
109         WRITE (3,10) ((G(I,J),J=1,NC),I=1,NF)
110         IT1=IT1+1
111         IF(IT1.LE.5) GO TO 1280
112         GO TO 3000
113    1380 TEST1=0.00
114         DO 2000 I=1,NF
115         DO 2000 J=1,NC
116         IF(G(I,J).EQ.  .000) GO TO 1400
117         TEST=DABS((G(I,J)-K(I,J))/G(I,J))
118         TEST1=DMAX1(TEST,TEST1)
119         GO TO 2000
120    1400 IF(G(I,J)-K(I,J).EQ.0.000) GO TO 2000
121         TEST1=1000.*GNSTOP
122    2000 K(I,J)=G(I,J)
123         WRITE (3,60) TEST1
124         WRITE (3,70) GNSTOP
125         IF(TEST1-GNSTOP.GT.0.000) GO TO 1260
126         WRITE (3,90)
127         WRITE (3,62) GF2
128         GO TO 8000
129    3000 WRITE (3,95)
130    8000 CONTINUE
131    9000 CONTINUE
132         STOP
133         END
```

```
      SUBROUTINE STRAM
C
C        COMPUTES THE STATE TRANSITION MATRIX
C
      COMMON M(3,3),MI(3,3),RC(3),AHAT(3,3),T,K(3,3),G(3,3),QHAT(3,3),
     1       FT(3,3),FFTI(3,3),DFT(3,3,3,3),FDFTI(3,3,3,3),
     2       D2FT(3,3,3,3,3,3),DFDFTI(3,3,3,3,3,3),FD2FTI(3,3,3,3,3,3),
     3       VW(9),VGRAD(9,9),GRADI(9,9),B(3,3),F(3,3),R(3,3),NS,NC,NF
      DIMENSION AAAA(9),RR(3),RI(3),ASQR(3,3),ASQ2(3,3),XR(3),XI(3),VR(3
     1          ),VI(3),IANA(3),IRCW(3,2),VRN(3),VIN(3),W(3,4)
      COMPLEX*16 M,MI,RC
      COMPLEX*16 DCMPLX,DCONJG
      DOUBLE PRECISION AHAT,T,K,G,QHAT,FT,FFTI,DFT,FDFTI,D2FT,DFDFTI,
     1                 FD2FTI,VW,VGRAD,GRADI
      DOUBLE PRECISION AAAA,RR,RI,ASQR,ASQ2,XR,XI,VR,VI,VRN,VIN,W,VECMGR
     1                 ,VECMGI,VECMGS,SW1
   10 FORMAT (2D18.7)
   30 FORMAT (//T3,'SYSTEM EIGENVALUES')
      CALL VECT(AHAT,AAAA,NS)
      CALL HSBG(NS,AAAA,NS)
      CALL ATEIG(NS,AAAA,RR,RI,IANA,NS)
      WRITE (3,30)
      WRITE (3,10) (RR(I),RI(I),I=1,NS)
      CALL MSC(AAAA,NS,ASQR)
      DO 100 I=1,NS
      RC(I)=DCMPLX(RR(I),RI(I))
      DO 100 J=1,NS
  100 ASQ2(I,J)=ASQR(I,J)
      CALL EIGVEC(3,AHAT,ASQR,W,IRCW,XR,XI,VR,VI,RR(1),RI(1),NS,NS,0,
     1            SW1,ITER,DIF,2)
      VECMGR=0.0D0
      VECMGI=0.0D0
      DO 110 I=1,NS
      VECMGR=VECMGR+VR(I)*XR(I)-VI(I)*XI(I)
  110 VECMGI=VECMGI+VR(I)*XI(I)+VI(I)*XR(I)
      VECMGS=VECMGR*VECMGR+VECMGI*VECMGI
      DO 120 I=1,NS
      VRN(I)=(VR(I)*VECMGR+VI(I)*VECMGI)/VECMGS
      VIN(I)=(VI(I)*VECMGR-VR(I)*VECMGI)/VECMGS
      M(I,1)=DCMPLX(XR(I),XI(I))
  120 MI(1,I)=DCMPLX(VRN(I),VIN(I))
      DO 130 KOUNT=2,NS
      KOUNT1=KOUNT-1
      IF(RR(KOUNT)-RR(KOUNT1)) 200,140,200
  140 CONTINUE
      DO 150 I=1,NS
      M(I,KOUNT)=DCONJG(M(I,KOUNT1))
  150 MI(KOUNT,I)=DCONJG(MI(KOUNT1,I))
      GO TO 1000
  200 DO 210 I=1,NS
      DO 210 J=1,NS
  210 ASQR(I,J)=ASQ2(I,J)
      CALL EIGVEC(3,AHAT,ASQR,W,IRCW,XR,XI,VR,VI,RR(KOUNT),RI(KOUNT),NS,
     1            NS,0,SW1,ITER,DIF,2)
      VECMGR=0.0D0
```

```
178           VECMGI=0.00.
179           DO 220 I=1,NS
180           VECMGR=VECMGR+VR(I)*XR(I)-VI(I)*XI(I)
181       220 VECMGI=VECMGI+VR(I)*XI(I)+VI(I)*XR(I)
182           VECMGS=VECMGR*VECMGR+VECMGI*VECMGI
183           DO 230 I=1,NS
184           VRN(I)=(VR(I)*VECMGR+VI(I)*VECMGI)/VECMGS
185           VIN(I)=(VI(I)*VECMGR-VR(I)*VECMGI)/VECMGS
186           M(I,KOUNT)=CCMPLX(XR(I),XI(I))
187       230 MI(KOUNT,I)=CCMPLX(VRN(I),VIN(I))
188      1000 CONTINUE
189           RETURN
190           END
```

```
      SUBROUTINE FEFA
C
C     COMPUTES REQUIRED FUNCTIONS OF THE STATE TRANSITION MATRIX
C
      COMMON M(3,3),MI(3,3),RC(3),AHAT(3,3),T,K(3,3),G(3,3),GHAT(3,3),
     1       FT(3,3),FFTI(3,3),DFT(3,3,3,3),FDFTI(3,3,3,3),
     2       D2FT(3,3,3,3,3,3),DFDFTI(3,3,3,3,3,3),FD2FTI(3,3,3,3,3,3),
     3       VW(3),VGRAD(9,9),GRADI(9,9),B(3,3),F(3,3),R(3,3),NS,NC,NF
      DIMENSION D1(3,3,3,3,3,3),D2(3,3,3,3,3,3)
      COMPLEX*16 EX1(3),EX2(3,3),R1(3,3),R2(3,3),DUM1,DUM2,DUM3,DUM4,
     1           DUM5,DUM6,DUM7
      COMPLEX*16 M,MI,RC
      COMPLEX*16 CDEXP
      DOUBLE PRECISION AHAT,T,K,G,GHAT,FT,FFTI,DFT,FDFTI,D2FT,DFDFTI,
     1                 FD2FTI,VW,VGRAL,GRADI
      DOUBLE PRECISION D1,D2
      NS1=NS-1
      NF1=NF-1
      NC1=NC-1
      DO 100 I=1,NS
      EX1(I)=CDEXP(RC(I)*T)
      DO 100 J=1,NS
      EX2(I,J)=CDEXP((RC(I)+RC(J))*T)
      R1(I,J)=RC(I)+RC(J)
  100 R2(I,J)=RC(I)-RC(J)
      DO 1000 I=1,NS
      DO 1000 J=1,NS
      FT(I,J)=0.0D0
      FFTI(I,J)=0.0D0
      DO 300 K1=1,NF
      DO 300 K2=1,NC
      DFT(I,J,K1,K2)=0.0D0
      FDFTI(I,J,K1,K2)=0.0D0
      DO 300 K3=1,NF
      DO 300 K4=1,NC
      D2FT(I,J,K1,K2,K3,K4)=0.0D0
      DFDFTI(I,J,K1,K2,K3,K4)=0.0D0
  300 FD2FTI(I,J,K1,K2,K3,K4)=0.0 D0
      DO 1000 N1=1,NS
      FT(I,J)=FT(I,J)+M(I,N1)*MI(N1,J)*EX1(N1)
      DO 1000 N2=1,NS
      DO 1000 N3=1,NS
      DUM1=M(I,N1)*MI(N1,N2)*M(J,N3)
      FFTI(I,J)=FFTI(I,J)+DUM1*MI(N3,N2)*(EX2(N1,N3)-1.)/R1(N1,N3)
      DO 600 K1=1,NF
      DO 600 K2=1,NC
      DUM2=M(I,N1)*MI(N1,N2)*B(N2,K2)*M(K1,N3)
      IF(N1-N3) 320,310,320
  310 DFT(I,J,K1,K2)=DFT(I,J,K1,K2)-DUM2*MI(N3,J)*T*EX1(N1)
      GO TO 330
  320 DFT(I,J,K1,K2)=DFT(I,J,K1,K2)-DUM2*MI(N3,J)*(EX1(N3)-EX1(N1))/R2(N
     13,N1)
  330 DO 600 N4=1,NS
      DO 600 N5=1,NS
      DUM3=DUM1*MI(N3,N4)*B(N4,K2)*M(K1,N5)*MI(N5,N2)
```

```
237          IF(N3-N5) 350,340,350
238     340  FDFTI(I,J,K1,K2)=FDFTI(I,J,K1,K2)-DUM3*(EX2(N1,N3)*(R1(N1,N3)*T-1.
          1)+1.)/(R1(N1,N3)*R1(N1,N3))
239          GO TO 360
240     350  FDFTI(I,J,K1,K2)=FDFTI(I,J,K1,K2)-DUM3*((EX2(N1,N5)-1.)/R1(N1,N5)-
          1(EX2(N1,N3)-1.)/R1(N1,N3))/R2(N5,N3)
241     360  DO 600 K3=1,NF
242          DO 600 K4=1,NC
243          DUM4=2.*DUM2*MI(N3,N4)*B(N4,K4)*M(K3,N5)*MI(N5,J)
244          IF(N3-N5) 420,370,420
245     370  IF(N1-N3) 390,380,390
246     380  D2FT(I,J,K1,K2,K3,K4)=D2FT(I,J,K1,K2,K3,K4)+DUM4*T*T*EX1(N1)/2.
247          GO TO 455
248     390  D2FT(I,J,K1,K2,K3,K4)=D2FT(I,J,K1,K2,K3,K4)+DUM4*((R2(N3,N1)*T-1.)
          1*EX1(N3)+EX1(N1))/(R2(N3,N1)*R2(N3,N1))
249          GO TO 455
250     400  CONTINUE
251          IF(N1-N3) 420,410,420
252     410  D2FT(I,J,K1,K2,K3,K4)=D2FT(I,J,K1,K2,K3,K4)-DUM4*T*EX1(N1)/R2(N5,N
          11)
253          GO TO 430
254     420  D2FT(I,J,K1,K2,K3,K4)=D2FT(I,J,K1,K2,K3,K4)-DUM4*(EX1(N3)-EX1(N1))
          1/(R2(N3,N1)*R2(N5,N3))
255     430  IF(N1-N5) 440,450,440
256     440  D2FT(I,J,K1,K2,K3,K4)=D2FT(I,J,K1,K2,K3,K4)+DUM4*(EX1(N5)-EX1(N1))
          1/(R2(N5,N1)*R2(N5,N3))
257          GO TO 455
258     450  D2FT(I,J,K1,K2,K3,K4)=D2FT(I,J,K1,K2,K3,K4)+DUM4*T*EX1(N5)/R2(N5,N
          13)
259     455  DO 600 N6=1,NS
260          DO 600 N7=1,NS
261          DUM5=MI(N4,N5)*B(N5,K4)*M(K3,N6)*MI(N6,N7)
262          DUM6=DUM2*MI(N3,N7)*M(J,N4)*DUM5
263          DUM7=2.*M(I,N1)*MI(N1,N7)*M(J,N2)*MI(N2,N3)*B(N3,K2)*M(K1,N4)*DUM5
264          IF(N1-N3) 490,460,490
265     460  IF(N4-N6) 480,470,480
266     470  DFDFTI(I,J,K1,K2,K3,K4)=DFDFTI(I,J,K1,K2,K3,K4)+DUM6*((T*T*R1(N1,N
          14)*R1(N1,N4)-2.*T*R1(N1,N4)+2.)*EX2(N1,N4)-2.)/(R1(N1,N4)*R1(N1,N4
          2)*R1(N1,N4))
267          GO TO 520
268     480  DFDFTI(I,J,K1,K2,K3,K4)=DFDFTI(I,J,K1,K2,K3,K4)+DUM6*(((R1(N1,N6)*
          1T-1.)*EX2(N1,N6)+1.)/(R1(N1,N6)*R1(N1,N6))-((R1(N1,N4)*T-1.)*EX2(N
          21,N4)+1.)/(R1(N1,N4)*R1(N1,N4)))/R2(N6,K4)
269          GO TO 520
270     490  IF(N4-N6) 510,500,510
271     500  DFDFTI(I,J,K1,K2,K3,K4)=DFDFTI(I,J,K1,K2,K3,K4)+DUM6*(((R1(N3,N4)*
          1T-1.)*EX2(N3,N4)+1.)/(R1(N3,N4)*R1(N3,N4))-((R1(N1,N4)*T-1.)*EX2(N
          21,N4)+1.)/(R1(N1,N4)*R1(N1,N4)))/R2(N3,N1)
272          GO TO 520
273     510  DFDFTI(I,J,K1,K2,K3,K4)=DFDFTI(I,J,K1,K2,K3,K4)+DUM6*((EX2(N3,N6)-
          11.)/R1(N3,N6)-(EX2(N3,N4)-1.)/R1(N3,N4)-(EX2(N1,N6)-1.)/R1(N1,N6)+
          2(EX2(N1,N4)-1.)/R1(N1,N4))/(R2(N3,N1)*R2(N6,N4))
274     520  IF(N4-N6) 560,530,560
275     530  IF(N2-N4) 550,540,550
276     540  FD2FTI(I,J,K1,K2,K3,K4)=FD2FTI(I,J,K1,K2,K3,K4)+.5*DUM7*(EX2(N1,N2
```

```
          1)*(T*T*R1(N1,N2)*R1(N1,N2)-2.*T*R1(N1,N2)+2.)-2.)/(R1(N1,N2)*R1(N1
          2,N2)*R1(N1,N2))
277           GO TO 600
278       550 FD2FTI(I,J,K1,K2,K3,K4)=FD2FTI(I,J,K1,K2,K3,K4)+DUM7*((EX2(N1,N4)*
          1(R1(N1,N4)*T-1.)+1.)/(R1(N1,N4)*R1(N1,N4))-(EX2(N1,N4)-1.)/(R1(N1,
          2N4)*R2(N4,N2))+(EX2(N1,N2)-1.)/(R1(N1,N2)*R2(N4,N2)))/R2(N4,N2)
279           GO TO 600
280       560 IF(N2-N4) 580,570,580
281       570 FD2FTI(I,J,K1,K2,K3,K4)=FD2FTI(I,J,K1,K2,K3,K4)-DUM7*(EX2(N1,N2)*(
          1R1(N1,N2)*T-1.)+1.)/(R1(N1,N2)*R1(N1,N2)*R2(N6,N2))
282           GO TO 590
283       580 FD2FTI(I,J,K1,K2,K3,K4)=FD2FTI(I,J,K1,K2,K3,K4)-DUM7*((EX2(N1,N4)
          1-1.)/R1(N1,N4)-(EX2(N1,N2)-1.)/R1(N1,N2))/(R2(N4,N2)*R2(N6,N4))
284       590 IF(N2-N6) 594,592,594
285       592 FD2FTI(I,J,K1,K2,K3,K4)=FD2FTI(I,J,K1,K2,K3,K4)+DUM7*(EX2(N1,N6)*
          1(R1(N1,N6)*T-1.)+1.)/(R1(N1,N6)*R2(N6,N4)*R1(N1,N6))
286           GO TO 600
287       594 FD2FTI(I,J,K1,K2,K3,K4)=FD2FTI(I,J,K1,K2,K3,K4)+DUM7*((EX2(N1,N6)-
          11.)/R1(N1,N6)-(EX2(N1,N2)-1.)/R1(N1,N2))/(R2(N6,N2)*R2(N6,N4))
288       600 CONTINUE
289           DO 700 K1=1,NF
290           DO 700 K2=1,NC
291           DO 700 K3=1,NF
292           DO 700 K4=1,NC
293           D1(I,J,K1,K2,K3,K4)=D2FT(I,J,K1,K2,K3,K4)
294       700 D2(I,J,K1,K2,K3,K4)=FD2FTI(I,J,K1,K2,K3,K4)
295           DO 800 K1=1,NF
296           DO 800 K2=1,NC
297           DO 800 K3=1,NF
298           DO 800 K4=1,NC
299           D2FT(I,J,K1,K2,K3,K4)=(D1(I,J,K1,K2,K3,K4)+D1(I,J,K3,K4,K1,K2))/2.
300       800 FD2FTI(I,J,K1,K2,K3,K4)=(D2(I,J,K1,K2,K3,K4)+D2(I,J,K3,K4,K1,K2))/
          12.
301      1000 CONTINUE
302           RETURN
303           END
```

```
304         SUBROUTINE GAINFN(GF)
      C
      C           COMPUTES AVERAGE COST
      C
305         COMMON M(3,3),MI(3,3),RC(3),AHAT(3,3),T,K(3,3),G(3,3),CHAT(3,3),
           1       FT(3,3),FFTI(3,3),DFT(3,3,3,3),FDFTI(3,3,3,3),
           2       D2FT(3,3,3,3,3,3),FD2FT(3,3,3,3,3,3),FD2FTI(3,3,3,3,3,3),
           3       VW(9),VGRAD(9,9),GRADI(9,9),R(3,3),F(3,3),N(3,3),NS,NC,NF
306         COMPLEX*16 M,MI,RC
307         DOUBLE PRECISION AHAT,T,K,G,CHAT,FT,FFTI,DFT,FDFTI,D2FT,FD2FTI,
           1                 FD2FTI,VW,VGRAD,GRADI
308         DOUBLE PRECISION GF
309         GF=0.0D0
310         DO 1000 N1=1,NS
311         DO 1000 N2=1,NS
312         GF=GF+QHAT(N1,N2)*FFTI(N2,N1)
313         DO 1000 N3=1,NS
314    1000 GF=GF+F(N1,N2)*FT(N2,N3)*FT(N1,N3)
315         GF=GF/NS
316         RETURN
317         END
```

```
      SUBROUTINE NECCON
C
C        COMPUTES NECESSARY CONDITIONS
C
      COMMON M(3,3),MI(3,3),RC(3),AHAT(3,3),T,K(3,3),G(3,3),QHAT(3,3),
     1       FT(3,3),FFTI(3,3),DFT(3,3,3,3),FDFTI(3,3,3,3),
     2       D2FT(3,3,3,3,3,3),DFDFTI(3,3,3,3,3,3),FD2FTI(3,3,3,3,3,3),
     3       VW(9),VGRAD(9,9),GRADI(9,9),B(3,3),F(3,3),R(3,3),NS,NC,NF
      COMPLEX*16 M,MI,RC
      DOUBLE PRECISION AHAT,T,K,G,QHAT,FT,FFTI,DFT,FDFTI,D2FT,DFDFTI,
     1                 FD2FTI,VW,VGRAD,GRADI
   10 FORMAT (//T3,'NECESSARY CONDITIONS VECTOR')
   20 FORMAT (4D18.7)
      NFC=NF*NC
      DO 1100 I=1,NF
      DO 1100 J=1,NC
      IN=NF*(J-1)+I
      VW(IN)=0.0D0
      DO 1100 N1=1,NS
      DO 1000 N2=1,NS
      VW(IN)=VW(IN)+QHAT(N1,N2)*FDFTI(N2,N1,I,J)
      DO 1000 N3=1,NS
 1000 VW(IN)=VW(IN)+F(N1,N2)*FT(N2,N3)*DFT(N1,N3,I,J)
      DO 1100 N4=1,NC
 1100 VW(IN)=VW(IN)+R(J,N4)*K(N1,N4)*FFTI(N1,I)
      WRITE (3,10)
      WRITE (3,20) (VW(I),I=1,NFC)
      RETURN
      END
```

```
340         SUBROUTINE GRADNT
      C
      C           COMPUTES GRADIENT MATRIX
      C
341         COMMON M(3,3),MI(3,3),RC(3),AHAT(3,3),T,K(3,3),G(3,3),CHAT(3,3),
           1       FT(3,3),FFTI(3,3),DFT(3,3,3,3),FDFTI(3,3,3,3),
           2       D2FT(3,3,3,3,3,3),DFDFTI(3,3,3,3,3,3),FD2FTI(3,3,3,3,3,3),
           3       VW(9),VGRAD(9,9),GRADI(9,9),R(3,3),F(3,3),R(3,3),NS,NC,NF
342         COMPLEX*16 M,MI,RC
343         DOUBLE PRECISION AHAT,T,K,G,CHAT,FT,FFTI,DFT,FDFTI,D2FT,DFDFTI,
           1                 FD2FTI,VW,VGRAD,GRADI
344    10   FORMAT (//T3,'GRADIENT MATRIX')
345    20   FORMAT (4D18.7)
346         NFC=NF*NC
347         DO 1100 I=1,NF
348         DO 1100 J=1,NC
349         IN=NF*(J-1)+I
350         DO 1100 K1=1,NF
351         DO 1100 K2=1,NC
352         ID=NF*(K2-1)+K1
353         VGRAD(IN,IC)=R(J,K2)*FFTI(K1,I)
354         DO 1100 N1=1,NS
355         DO 1000 N2=1,NS
356         VGRAD(IN,IC)=VGRAD(IN,ID)+CHAT(N1,N2)*(DFDFTI(N2,N1,K1,K2,I,J)+FD2
           1FTI(N2,N1,K1,K2,I,J))
357         DO 1100 N3=1,NS
358    1000 VGRAD(IN,ID)=VGRAD(IN,ID)+F(N1,N2)*(DFT(N2,N3,K1,K2)*DFT(N1,N3,I,J
           2)+FT(N2,N3)*D2FT(N1,N3,K1,K2,I,J))
359         DO 1100 N4=1,NC
360    1100 VGRAD(IN,IC)=VGRAD(IN,ID)+R(K2,N4)*K(N1,N4)*(FDFTI(N1,K1,I,J)+
           1                 FDFTI(K1,N1,I,J)) + R(J,N4)*K(N1,N4)*
           2                 (FDFTI(N1,I,K1,K2)+FDFTI(I,N1,K1,K2))
361         WRITE (3,10)
362         WRITE (3,20) ((VGRAD(I,J),J=1,NFC),I=1,NFC)
363         RETURN
364         END
```

```
      SUBROUTINE INVERT(A,B,N)
C
C        INVERTS A TO GIVE B
C
      DIMENSION A(9,9),B(9,9)
      DOUBLE PRECISION A,B,C,D,X
      DOUBLE PRECISION DABS
      IF(N-1) 100,100,101
  100 B(1,1)=1.0D0/A(1,1)
      RETURN
  101 DO 102 I=1,N
      DO 102 J=1,N
  102 B(I,J)=0.0D0
      DO 103 I=1,N
  103 B(I,I)=1.0D0
C     PICK UP PIVOT ELEMENT
      DO 114 K=1,N
      L=K
      IF(N-K) 112,110,104
  104 I=K+1
      DO 106 JJ=I,N
      IF(DABS(A(JJ,K))-DABS(A(L,K)))106,106,105
  105 L=JJ
  106 CONTINUE
      IF(L-K) 107,110,107
C     PERFORM ROW INTERCHANGE
  107 DO 108 J=K,N
      C=A(K,J)
      A(K,J)=A(L,J)
  108 A(L,J)=C
      DO 109 J=1,N
      C=B(K,J)
      B(K,J)=B(L,J)
  109 B(L,J)=C
C     COLUMN ELIMINATION
  110 DO 114 I=1,N
      IF(K-I) 111,114,111
  111 D=A(I,K)/A(K,K)
      DO 112 J=K,N
  112 A(I,J)=A(I,J)-D*A(K,J)
      A(I,K)=0.0D0
      DO 113 J=1,N
  113 B(I,J)=B(I,J)-D*B(K,J)
  114 CONTINUE
C     SOLVE FOR INVERSE
      DO 115 J=1,N
      DO 115 I=1,N
      X=B(I,J)/A(I,I)
  115 B(I,J)=X
      RETURN
      END
```

```
      SUBROUTINE NEWRIT
C
C        PERFORMS NEWTON RAPHSON ITERATION
C
      COMMON M(3,3),MI(3,3),RC(3),AHAT(3,3),T,K(3,3),G(3,3),GHAT(3,3),
     1       FT(3,3),FFTI(3,3),CFT(3,3,3,3),FCFTI(3,3,3,3),
     2       C2FT(3,3,3,3,3,3),CFCFTI(3,3,3,3,3,3),FC2FTI(3,3,3,3,3,3),
     3       VW(9),VGRAD(9,9),GRADI(9,9),B(3,3),F(3,3),R(3,3),NS,NC,NF
      COMPLEX*16 M,MI,RC
      DOUBLE PRECISION AHAT,T,K,G,GHAT,FT,FFTI,CFT,FCFTI,C2FT,CFCFTI,
     1                 FC2FTI,VW,VGRAD,GRADI
      DO 1000 I=1,NF
      DO 1000 J=1,NC
      G(I,J)=K(I,J)
      IN=NF*(J-1)+I
      DO 1000 K1=1,NF
      DO 1000 K2=1,NC
      ID=NF*(K2-1)+K1
 1000 G(I,J)=G(I,J)-GRADI(IN,ID)*VW(ID)
      RETURN
      END
```

```
      SUBROUTINE VECT(AHAT,AAAA,NS)
C
C     CONVERTS AHAT TO SINGLE SUBSCRIPT FORM AAAA
C
      DIMENSION AHAT(3,3),AAAA(9)
      DOUBLE PRECISION AHAT,AAAA
      DO 100 J=1,NS
      DO 100 I=1,NS
      K=(J-1)*NS+I
  100 AAAA(K)=AHAT(I,J)
      RETURN
      END
```

```
      SUBROUTINE GAIN2(GF2)
C
C        COMPUTES AVERAGE COST FOR CONVERGENCE CHECK
C
      COMMON M(3,3),MI(3,3),RC(3),AHAT(3,3),T,K(3,3),G(3,3),QHAT(3,3),
     1       FT(3,3),FFTI(3,3),DFT(3,3,3,3),FDFTI(3,3,3,3),
     2       D2FT(3,3,3,3,3,3),DFDFTI(3,3,3,3,3,3),FD2FTI(3,3,3,3,3,3),
     3       VW(9),VGRAD(9,9),GRADI(9,9),B(3,3),F(3,3),R(3,3),NS,NC,NF
      COMPLEX*16 EX1(3),EX2(3,3),R1(3,3),DUM1
      COMPLEX*16 M,MI,RC
      COMPLEX*16 CDEXP
      DOUBLE PRECISION AHAT,T,K,G,QHAT,FT,FFTI,DFT,FDFTI,D2FT,DFDFTI,
     1                 FD2FTI,VW,VGRAD,GRADI
      DOUBLE PRECISION GF2
      DO 500 I=1,NS
      EX1(I)=CDEXP(RC(I)*T)
      DO 500 J=1,NS
      EX2(I,J)=CDEXP((RC(I)+RC(J))*T)
  500 R1(I,J)=RC(I)+RC(J)
      DO 1000 I=1,NS
      DO 1000 J=1,NS
      FT(I,J)=0.000
      FFTI(I,J)=0.000
      DO 1000 N1=1,NS
      FT(I,J)=FT(I,J)+M(I,N1)*MI(N1,J)*EX1(N1)
      DO 1000 N2=1,NS
      DO 1000 N3=1,NS
      DUM1=M(I,N1)*MI(N1,N2)*M(J,N3)
 1000 FFTI(I,J)=FFTI(I,J)+DUM1*MI(N3,N2)*(EX2(N1,N3)-1.)/R1(N1,N3)
      GF2=0.000
      DO 2000 N1=1,NS
      DO 2000 N2=1,NS
      GF2=GF2+QHAT(N1,N2)*FFTI(N2,N1)
      DO 2000 N3=1,NS
 2000 GF2=GF2+F(N1,N2)*FT(N2,N3)*FT(N1,N3)
      GF2=GF2/NS
      RETURN
      END
```

```
      SUBROUTINE HSBG(N,A,IA)
C
C     CONVERTS A TC UPPER HESSENBERG FORM
C
      DIMENSION A(9)
      DOUBLE PRECISION A,PIV,T,S
      DOUBLE PRECISION DABS
      L=N
      NIA=L*IA
      LIA=NIA-IA
   20 IF(L-3) 360,40,40
   40 LIA=LIA-IA
      L1=L-1
      L2=L1-1
      ISUB=LIA+L
      IPIV=ISUB-IA
      PIV=DABS(A(IPIV))
      IF(L-3) 90,90,50
   50 M=IPIV-IA
      DO 80 I=L,M,IA
      T=DABS(A(I))
      IF(T-PIV) 80,80,60
   60 IPIV=I
      PIV=T
   80 CONTINUE
   90 IF(PIV) 100,320,100
  100 IF(PIV-DABS(A(ISUB))) 180,180,120
  120 M=IPIV-L
      DO 140 I=1,L
      J=M+I
      T=A(J)
      K=LIA+I
      A(J)=A(K)
  140 A(K)=T
      M=L2-M/IA
      DO 160 I=L1,NIA,IA
      T=A(I)
      J=I-M
      A(I)=A(J)
  160 A(J)=T
  180 DO 200 I=L,LIA,IA
  200 A(I)=A(I)/A(ISUB)
      J=-IA
      DO 240 I=1,L2
      J=J+IA
      LJ=L+J
      DO 220 K=1,L1
      KJ=K+J
      KL=K+LIA
  220 A(KJ)=A(KJ)-A(LJ)*A(KL)
  240 CONTINUE
      K=-IA
      DO 300 I=1,N
      K=K+IA
      LK=K+L1
```

```
      S=A(LK)
      LJ=L-IA
      DO 280 J=1,L2
      JK=K+J
      LJ=LJ+IA
  280 S=S+A(LJ)*A(JK)*1.GDC
  300 A(LK)=S
      DO 310 I=L,LIA,IA
  310 A(I)=0.GDC
  320 L=L1
      GO TC 20
  360 RETURN
      END
```

```
      SUBROUTINE ATEIG(M,A,RR,RI,IANA,IA)
C
C     COMPUTES ROOTS OF UPPER HESSENBERG MATRIX A
C
      DIMENSION A(9),RR(3),RI(3),PRR(2),PRI(2),IANA(3)
      DOUBLE PRECISION E7,E6,E10,DELTA,PRR,PRI,PAN,PAN1,R,S,T,A,U,V,RR,
     1                 RI,RMOD,EPS,D,G1,G2,G3,CAP,PSI1,PSI2,ALPHA,ETA
      DOUBLE PRECISION DABS,DSQRT,DMAX1
      INTEGER P,P1,Q
      E7=1.0D-8
      E6=1.0D-6
      E10=1.0D-10
      DELTA=0.500
      MAXIT=30
      N=M
   20 N1=N-1
      IN=N1*IA
      NN=IN+N
      IF(N1) 30,1300,30
   30 NP=N+1
      IT=0
      DO 40 I=1,2
      PRR(I)=0.0D0
   40 PRI(I)=0.0D0
      PAN=0.0D0
      PAN1=0.0D0
      R=0.0D0
      S=0.0D0
      N2=N1-1
      IN1=IN-IA
      NN1=IN1+N
      N1N=IN+N1
      N1N1=IN1+N1
   60 T=A(N1N1)-A(NN)
      U=T*T
      V=4.0D0*A(N1N)*A(NN1)
      IF(DABS(V)-U*E7) 100,100,65
   65 T=U+V
      IF(DABS(T)-DMAX1(U,DABS(V))*E6) 67,67,68
   67 T=0.0D0
   68 U=(A(N1N1)+A(NN))/2.0D0
      V=DSQRT(DABS(T))/2.0D0
      IF(T)140,70,70
   70 IF(U) 80,75,75
   75 RR(N1)=U+V
      RR(N)=U-V
      GO TO 130
   80 RR(N1)=U-V
      RR(N)=U+V
      GO TO 130
  100 IF(T)120,110,110
  110 RR(N1)=A(N1N1)+V
      RR(N)=A(NN)
      GO TO 130
  120 RR(N1)=A(NN)
```

```
      RR(N)=A(N1N1)
  130 RI(N)=0.0D0
      RI(N1)=0.0D0
      RI(N1)=0.0
      GO TO 160
  140 RR(N1)=U
      RR(N)=U
      RI(N1)=V
      RI(N)=-V
  160 IF(N2)1280,1280,180
  180 N1N2=N1N1-IA
      RMOD=RR(N1)*RR(N1)+RI(N1)*RI(N1)
      EPS=E1D*DSQRT(RMOD)
      IF(DABS(A(N1N2))-EPS) 1280,1280,240
  240 IF(DABS(A(NN1))-E1D*DABS(A(NN))) 1300,1300,250
  250 IF(DABS(PAN1-A(N1N2))-DABS(A(N1N2))*E6) 1240,1240,260
  260 IF(DABS(PAN-A(NN1))-DABS(A(NN1))*E6) 1240,1240,300
  300 IF(IT-MAXIT) 320,1240,1240
  320 J=1
      DO 360 I=1,2
      K=NP-I
      IF(DABS(RR(K)-PRR(I))+DABS(RI(K)-PRI(I))-DELTA*(DABS(RR(K))
     1     +DABS(RI(K)))) 340,360,360
  340 J=J+I
  360 CONTINUE
      GO TO (440,460,460,480),J
  440 R=0.0D0
      S=0.0D0
      GO TO 500
  460 J=N+2-J
      R=RR(J)*RR(J)
      S=RR(J)+RR(J)
      GO TO 500
  480 R=RR(N)*RR(N1)-RI(N)*RI(N1)
      S=RR(N)+RR(N1)
  500 PAN=A(NN1)
      PAN1=A(N1N2)
      DO 520 I=1,2
      K=NP-I
      PRR(I)=RR(K)
  520 PRI(I)=RI(K)
      P=N2
      IF(N-3)600,600,525
  525 IPI=N1N2
      DO 580 J=2,N2
      IPI=IPI-IA-1
      IF(DABS(A(IPI))-EPS) 600,600,530
  530 IPIP=IPI+IA
      IPIP2=IPIP+IA
      C=A(IPIP)*(A(IPIP)-S)+A(IPIP2)*A(IPIP+1)+R
      IF(D)540,560,540
  540 IF(DABS(A(IPI)*A(IPIP+1))*(DABS(A(IPIP)+A(IPIP2+1)-S)+DABS(A(IPIP2
     1     +2)))-DABS(C)*EPS) 620,620,560
  560 P=N1-J
  580 CONTINUE
```

```
      600 Q=P
          GO TO 680
      620 P1=P-1
          Q=P1
          IF(P1-1)680,680,650
      650 DO 660 I=2,P1
          IPI=IPI-IA-1
          IF(DABS(A(IPI))-EPS) 680,680,660
      660 Q=Q-1
      680 II=(P-1)*IA+P
          DO 1220 I=P,N1
          III=II-IA
          IIP=II+IA
          IF(I-P)720,700,720
      700 IPI=II+1
          IPIP=IIP+1
          G1=A(II)*(A(II)-S)+A(IIP)*A(IPI)+R
          G2=A(IPI)*(A(IPIP)+A(II)-S)
          G3=A(IPI)*A(IPIP+1)
          A(IPI+1)=0.0D0
          GO TO 780
      720 G1=A(III)
          G2=A(III+1)
          IF(I-N2)740,740,760
      740 G3=A(III+2)
          GO TO 780
      760 G3=0.0D0
      780 CAP=DSQRT(G1*G1+G2*G2+G3*G3)
          IF(CAP)800,860,800
      800 IF(G1)820,840,840
      820 CAP=-CAP
      840 T=G1+CAP
          PSI1=G2/T
          PSI2=G3/T
          ALPHA=2.0D0/(1.0D0+PSI1*PSI1+PSI2*PSI2)
          GO TO 880
      860 ALPHA=2.0D0
          PSI1=0.0D0
          PSI2=0.0D0
      880 IF(I-Q)900,960,900
      900 IF(I-P)920,940,920
      920 A(III)=-CAP
          GO TO 960
      940 A(III)=-A(III)
      960 IJ=II
          DO 1040 J=I,N
          T=PSI1*A(IJ+1)
          IF(I-N1)980,1000,1000
      980 IP2J=IJ+2
          T=T+PSI2*A(IP2J)
     1000 ETA=ALPHA*(T+A(IJ))
          A(IJ)=A(IJ)-ETA
          A(IJ+1)=A(IJ+1)-PSI1*ETA
          IF(I-N1)1020,1040,1040
     1020 A(IP2J)=A(IP2J)-PSI2*ETA
```

```
      1040 IJ=IJ+IA
           IF(I-N1)1080,1060,1060
      1060 K=N
           GO TO 1100
      1080 K=I+2
      1100 IP=IIP-I
           DO 1180 J=Q,K
           JIP=IP+J
           JI=JIP-IA
           T=PSI1*A(JIP)
           IF(I-N1)1120,1140,1140
      1120 JIP2=JIP+IA
           T=T+PSI2*A(JIP2)
      1140 ETA=ALPHA*(T+A(JI))
           A(JI)=A(JI)-ETA
           A(JIP)=A(JIP)-ETA*PSI1
           IF(I-N1)1160,1180,1180
      1160 A(JIP2)=A(JIP2)-ETA*PSI2
      1180 CONTINUE
           IF(I-N2)1200,1220,1220
      1200 JI=II+3
           JIP=JI+IA
           JIP2=JIP+IA
           ETA=ALPHA*PSI2*A(JIP2)
           A(JI)=-ETA
           A(JIP)=-ETA*PSI1
           A(JIP2)=A(JIP2)-ETA*PSI2
      1220 II=IIP+1
           IT=IT+1
           GO TO 60
      1240 IF(DABS(A(NN1))-DABS(A(N1N2)))1300,1280,1280
      1280 IANA(N)=0
           IANA(N1)=2
           N=N2
           IF(N2)1400,1400,20
      1300 RR(N)=A(NN)
           RI(N)=0.000
           IANA(N)=1
           IF(N1)1400,1400,1320
      1320 N=N1
           GO TO 20
      1400 RETURN
           END
```

```
      SUBRCUTINE MSC(AAAA,NS,ASQR)
C
C     COMPLTES ASQR = AAAA*AAAA
C
      DIMENSION AAAA(9),ASQR(3,3)
      DOUBLE PRECISION AAAA,ASQR
      DO 100 I=1,NS
      DO 100 J=1,NS
      ASQR(I,J)=0.000
      DO 100 K=1,NS
      K1=NS*(J-1)+K
      K2=NS*(K-1)+I
  100 ASQR(I,J)=ASQR(I,J)+AAAA(K1)*AAAA(K2)
      RETURN
      END
```

```
      SUBROUTINE EIGVEC(IVC,  A, B, N, IROW, XR, XI, VR, VI, RCOTRE,
     1   RCOTIE, NE, AMAX, T2, SW1, COUNTE, ERR,MMM)
C        SUBROUTINE TO FIND THE EIGENVECTORS OF A NON-SYMMETRIC MATRIX
C        BY A MODIFIED WILKINSON'S INVERSE ITERATION METHOD.
C        CONTROL IVC CODE IS
C             1  FIND ONLY THE REGULAR EIGENVECTORS   (A X = LAMBDA X)
C             2  FIND ONLY THE TRANSPOSED EIGENVECTORS (AT V = LAMBDA V)
C             3  FIND BOTH TYPES OF EIGENVECTORS.
      DIMENSION A(3,3),B(3,3),W(3,2),XR(3),XI(3),VR(3),VI(3),IROW(3,2)
      DOUBLE PRECISION RCOTR,RCOTI,RCOTRE,RCOTIE,TEMP,TEMP2,AMAX,C1,C2,
     1                 SW1,W,XR,XI,VR,VI,B,ZERO,COMPR,A
      DOUBLE PRECISION DABS,DSIGN,DSQRT,BMAX1
      INTEGER COUNT, COUNTE, T2
      IO1=1
      IO3=3
      RCOTR = RCOTRE
      RCOTI = RCOTIE
      N = NE
      MM = MMM - 1
      N1 = N - 1
      NP1 = N + 1
      IVC1 = IVC - 1
      IVC2 = IVC1 - 1
      COUNT = 1
      DO 400 I=1,N
      W(I,1)=0.CDC
      XR(I)=0.CDC
  400 CONTINUE
      CLIM = 1.E-4
      IF(RCOTI) 1, 60, 1
C
C        COMPLEX EIGENVALUE.
C
    1 TEMP = - RCOTR - RCOTR
      ISW = 2
      TEMP2=RCOTR*RCOTR+RCOTI*RCOTI
      JJ = 300
      DO 606 I = 1, N
      IF(T2) 600, 603, 600
  600 DO 602 J = 1, N
      JJ = JJ + 1
      IF(JJ - 251) 602, 601, 601
  601 JJ = 1
      READ (T2) (W(LL,1), LL = 1,250)
  602 B(I,J) = A(I,J)*TEMP + W(JJ,1)
      GO TO 605
  603 DO 604 J = 1, N
  604 B(I,J) = A(I,J)*TEMP + B(I,J)
  605 B(I,I) = B(I,I) + TEMP2
  606 A(I,I) = A(I,I) - RCOTR
      IF(T2 .NE. 0) REWIND T2
      GO TO 700
  607 IF(ICC) 622, 608, 622
C
C        MATRIX SINGULAR.
```

```
      C                                                                 ES
  784   622 IF(IVC2) 623, 625, 623                                      ES
  785   623 DO 624 LL = 1, N                                            ES
  786       W(LL,2)=0.0D0
  787   624 XI(LL)=0.0D0
  788       IF(IVC1) 625, 514, 625                                      ES
  789   625 DO 626 LL = 1, N                                            ES
  790       W(LL,4)=0.0D0
  791   626 VI(LL)=0.0D0
  792       GO TO 511                                                   ES
      C                                                                 ES
      C        MATRIX NOT SINGULAR.                                     ES
      C                                                                 ES
  793   608 DO 609 LL = 1, N                                            ES
  794       W(LL,1)=1.0D0
  795       W(LL,2)=1.0D0
  796       W(LL,3)=1.0D0
  797   609 W(LL,4)=1.0D0
  798   699 IF(IVC2) 610, 612, 610                                      ES
  799   610 DO 611 I = 1, N                                             ES
  800       I2 = IROW(I,2)                                              ES
  801       XI(I2) = W(I,1)*RGCTI                                       ES
  802       DO 611 J = 1, N                                             ES
  803   611 XI(I2) = XI(I2) + A(I,J)*W(J,2)                             ES
  804       IF(IVC1) 612, 500, 612                                      ES
  805   612 DO 613 I = 1, N                                             ES
  806       VI(I) = W(I,3)*RGCTI                                        ES
  807       DO 613 J = 1,N                                              ES
  808   613 VI(I) = VI(I) + A(J,I)*W(J,4)                               ES
  809       GO TO 499                                                   ES
  810   615 CERR = 0.0                                                  ES
  811       DCERR=0.0D0
  812       IF(IVC2) 616, 619, 616                                      ES
  813   616 DO 618 I = 1, N                                             ES
  814       XR(I) = -W(I,2)                                             ES
  815       DO 617 J = 1, N                                             ES
  816   617 XR(I) = XR(I) + A(I,J)*XI(J)                                ES
  817   618 XR(I) = XR(I)/RGCTI                                         ES
  818       IF(IVC1) 619, 633, 619                                      ES
  819   619 DO 621 I = 1, N                                             ES
  820       VR(I) = -W(I,4)                                             ES
  821       DO 620 J = 1, N                                             ES
  822   620 VR(I) = VR(I) + A(J,I)*VI(J)                                ES
  823   621 VR(I) = VR(I)/RGCTI                                         ES
      C                                                                 ES
      C        SEARCH VECTORS FOR LARGEST ELEMENT AND NORMALIZE.        ES
      C                                                                 ES
  824   627 AMAX=0.0D0
  825       DO 629 L = 1, N                                             ES
  826       TEMP = VR(L)**2 + VI(L)**2                                  ES
  827       IF(TEMP - AMAX) 629, 629, 628                               ES
  828   628 AMAX = TEMP                                                 ES
  829       I2 = L                                                      ES
  830   629 CONTINUE                                                    ES
  831       C1 = VR(I2)/AMAX                                            ES
```

```
832         C2 = -VI(I2)/AMAX
833         DO 630 L = 1, N
834         TEMP = VI(L)
835         VI(L) = VR(L)*C2 + TEMP*C1
836      630 VR(L) = VR(L)*C1 - TEMP*C2
837         IF(COUNT .EQ. 1) GO TO 632
838         DO 631 LL = 1, N
839      631  DCERR=DMAX1(DCERR,DABS(VR(LL)-W(LL,3)),DABS(VI(LL)-W(LL,4)))
840      632 IF(IVC2) 633, 636, 633
841      633  AMAX=0.000
842         DO 635 L = 1, N
843         TEMP = XR(L)**2 + XI(L)**2
844         IF(TEMP - AMAX) 635, 635, 634
845      634 AMAX = TEMP
846         I2 = L
847      635 CONTINUE
848         C1 = XR(I2)/AMAX
849         C2 = -XI(I2)/AMAX
850         DO 636 L = 1, N
851         TEMP = XI(L)
852         XI(L) = XR(L)*C2 + TEMP*C1
853      636 XR(L) = XR(L)*C1 - TEMP*C2
854         IF(COUNT .EQ. 1) GO TO 646
855         DO 637 LL = 1, N
856      637  DCERR=DMAX1(DCERR,DABS(XR(LL)-W(LL,1)),DABS(XI(LL)-W(LL,2)))
         C
         C        TEST FOR CONVERGENCE.
         C
857      638 IF(COUNT .EQ. 1) GO TO 646
858         CERR=DCERR
859         IF(CERR .GE. 1.0E-4) GO TO 639
860         IF(CERR .GE. CLIM) GO TO 648
861         CLIM = CERR
862         IF(CLIM .LE. 1.0E-8) GO TO 648
863      639 IF(COUNT .GE. 15) GO TO 68
864      647 COUNT = COUNT + 1
865         IF(RCOTI) 642, 673, 642
866      642 IF(IVC2) 643, 644, 643
867      643 DO 641 LL = 1, N
868         W(LL,1) = XR(LL)
869      641 W(LL,2) = XI(LL)
870         IF(IVC1) 644, 610, 644
871      644 DO 645 LL = 1, N
872         W(LL,3) = VR(LL)
873      645 W(LL,4) = VI(LL)
874         GO TO 699
875      646 CERR = 0.0
876         DCERR=0.000
877         IF(ICC) 648, 647, 648
878      648 ERR = CERR
879         COUNTE = COUNT
880         IF(RCOTI) 667, 668, 667
881      667 DO 649 I = 1, N
882      649 A(I,I) = A(I,I) + RCOTR
883         RETURN
```

```
884      68 PRINT 101, RCCTR, RCCTI, CERR                                  ES
885         GO TO 648                                                      ES
      C                                                                    ES
      C        REAL EIGENVECTORS.                                          ES
      C                                                                    ES
886      60 ISW = 1                                                        ES
887         DO 651 I = 1, N                                                ES
888         DO 650 J = 1, N                                                ES
889     650 B(I,J) = A(I,J)                                                ES
890     651 B(I,I) = B(I,I) - RCCTR                                        ES
891         GO TO 700                                                      ES
892     652 IF(ICC) 680, 685, 680                                          ES
      C                                                                    ES
      C        SINGULAR MATRIX.                                            ES
      C                                                                    ES
893     680 IF(IVC2) 681, 683, 681                                         ES
894     681 DO 682 L = 1, N                                                ES
895     682 XI(L)=0.0D0
896         IF(IVC1) 683, 514, 683                                         ES
897     683 DO 684 L = 1, N                                                ES
898     684 VI(L)=0.0D0
899         GO TO 511                                                      ES
      C                                                                    ES
      C        MATRIX NOT SINGULAR.                                        ES
      C                                                                    ES
900     685 IF(IVC2) 653, 656, 653                                         ES
901     653 DO 654 L = 1, N                                                ES
902     654 XI(L)=1.0D0
903         IF(IVC1) 656, 500, 656                                         ES
904     656 DO 657 L = 1, N                                                ES
905     657 VI(L)=1.0D0
906         GO TO 459                                                      ES
      C                                                                    ES
      C        NORMALIZE REAL VECTORS.                                     ES
      C                                                                    ES
907     655 CERR = 0.0                                                     ES
908         DCERR=0.0D0
909         IF(IVC2) 658, 662, 658                                         ES
910     658 C1=0.0D0
911         C2=0.0D0
912         DO 660 L = 1, N                                                ES
913         TEMP=DABS(XI(L))
914         IF(TEMP - C1) 660, 660, 659                                    ES
915     659 C1 = TEMP                                                      ES
916         C2 = XI(L)                                                     ES
917     660 CONTINUE                                                       ES
918         DO 661 L = 1, N                                                ES
919         XI(L) = XI(L)/C2                                               ES
920         DCERR=DMAX1(DCERR,DABS(XI(L)-XR(L)))
921     661 XR(L) = XI(L)                                                  ES
922         IF(IVC1) 662, 638, 662                                         ES
923     662 C2=0.0D0
924         C1=0.0D0
925         DO 664 L = 1, N                                                ES
926         TEMP=DABS(VI(L))
```

```
927          IF(TEMP - C1) 664, 664, 663                                    ES
928      663 C1 = TEMP                                                      ES
929          C2 = VI(L)                                                     ES
930      664 CONTINUE                                                       ES
931          DO 665 LL = 1, N                                               ES
932          VI(LL) = VI(LL)/C2                                             ES
933          DCERR=DMAX1(DCERR,DABS(VI(LL)-W(LL,1)))
934          W(LL,1)=VI(LL)
935      665 VR(LL)=W(LL,1)
936          GO TO 638                                                      ES
937      668 IF(IVC2) 669, 671, 669                                         ES
938      669 DO 670 L = 1, N                                                ES
939      670 XI(L)=0.0D0
940          IF(IVC1) 671, 70, 671                                          ES
941      671 DO 672 L = 1, N                                                ES
942      672 VI(L)=0.0D0
943       70 RETURN                                                         ES
944      673 IF(IVC2) 674, 502, 674                                         ES
945      674 DO 675 I = 1, N                                                ES
946          I2 = IRCW(I,2)                                                 ES
947      675 XI(I2) = XR(I)                                                 ES
948          GO TO 500                                                      ES
         C                                                                  ES
         C        BACK SUBSTITUTION SECTION.                                ES
         C                                                                  ES
949      499 IF(IVC2) 500, 502, 500                                         ES
950      500 DO 501 I = 2, N                                                ES
951          I1 = I - 1                                                     ES
952          DO 501 J = 1, I1                                               ES
953      501 XI(I) = XI(I) - B(I,J)*XI(J)                                   ES
954      511 IF(IVC1) 502, 514, 502                                         ES
955      502 DO 510 I = 1, N                                                ES
956          I1 = I - 1                                                     ES
957          IF(I1) 503, 505, 503                                           ES
958      503 DO 504 J = 1, I1                                               ES
959      504 VI(I) = VI(I) - B(J,I)*VI(J)                                   ES
960          IF(ICC) 505, 506, 505                                          ES
961      505 IF(B(I,I)) 506, 507, 506                                       ES
962      506 VI(I) = VI(I)/B(I,I)                                           ES
963          GO TO 510                                                      ES
964      507 IF(VI(I)) 508, 509, 508                                        ES
965      508 VI(I) = VI(I)*1.0E+15                                          ES
966          GO TO 510                                                      ES
967      509 VI(I) = 1.0                                                    ES
968      510 CONTINUE                                                       ES
969          IF(IVC2) 514, 525, 514                                         ES
970      514 DO 522 I = 1, N                                                ES
971          IR = NP1 - I                                                   ES
972          IF(I - 1) 515, 517, 515                                        ES
973      515 I2 = IR + 1                                                    ES
974          DO 516 J = I2, N                                               ES
975      516 XI(IR) = XI(IR) - B(IR,J)*XI(J)                                ES
976          IF(ICC) 517, 518, 517                                          ES
977      517 IF(B(IR,IR)) 518, 519, 518                                     ES
978      518 XI(IR) = XI(IR)/B(IR,IR)                                       ES
```

```
      GO TO 522                                                         ES
  519 IF(XI(IR)) 520, 521, 520                                          ES
  520 XI(IR) = XI(IR)*1.0E+15                                           ES
      GO TO 522                                                         ES
  521 XI(IR)=1.0D0
  522 CONTINUE                                                          ES
      IF(IVC1) 525, 529, 525                                            ES
  525 DO 526 I = 2, N                                                   ES
      IR = NP1 - I                                                      ES
      I2 = IR + 1                                                       ES
      DO 526 J = I2, N                                                  ES
  526 VI(IR) = VI(IR) - B(J,IR)*VI(J)                                   ES
      DO 527 L = 1, N                                                   ES
      I2 = IROW(L,1)                                                    ES
  527 VR(I2) = VI(L)                                                    ES
      DO 528 L = 1, N                                                   ES
  528 VI(L) = VR(L)                                                     ES
  529 IF(RCOTI) 615, 656, 615                                           ES
C                                                                       ES
C        FACTOR MATRIX.                                                 ES
C                                                                       ES
  700 ICC = 0                                                           ES
      SW1=1.0D72
      DO 701 LL = 1, N                                                  ES
  701 IROW(LL,1) = LL                                                   ES
      DO 708 K = 1, N1                                                  ES
      AMAX=DABS(B(K,K))
      IMAX = K                                                          ES
      K1 = K + 1                                                        ES
      DO 702 I = K1, N                                                  ES
      IF(AMAX.GT.CABS(B(I,K))) GO TO 702
      AMAX=DABS(B(I,K))
      IMAX = I                                                          ES
  702 CONTINUE                                                          ES
      IF(AMAX .LT. SW1) SW1 = AMAX                                      ES
      IF(AMAX.GE.1.0D-25) GO TO 723
      B(K,K)=0.0D0
      ICC = ICC + 1                                                     ES
      GO TO 708                                                         ES
  723 IF(IMAX .EQ. K) GO TO 704                                         ES
      DO 703 J = 1, N                                                   ES
      AMAX = B(K,J)                                                     ES
      B(K,J) = B(IMAX,J)                                                ES
  703 B(IMAX,J) = AMAX                                                  ES
      I2 = IROW(K,1)                                                    ES
      IROW(K,1) = IROW(IMAX,1)                                          ES
      IROW(IMAX,1) = I2                                                 ES
  704 DO 707 I = K1, N                                                  ES
      IF(B(I,K)) 705, 707, 705                                          ES
  705 B(I,K) = B(I,K)/B(K,K)                                            ES
      DO 706 J = K1, N                                                  ES
  706 B(I,J) = B(I,J) - B(K,J)*B(I,K)                                   ES
  707 CONTINUE                                                          ES
  708 CONTINUE                                                          ES
      AMAX=DA[illegible]
```

```
1031          IF(AMAX-1.0D-25) 712,712,713
1032      712 B(N,N)=0.0D0
1033          SW1=0.0D0
1034          ICC = ICC + 1                                                    ES\
1035          GO TO 709                                                        ES\
1036      713 IF(AMAX .LT. SW1) SW1 = AMAX                                     ES\
1037      709 IF(ICC .LE. ISW) GO TO 710                                       ES\
1038          IF(MM) 1050,1050,1051
1039     1051 WRITE(IO3,102) ICC
1040          COUNTE = 0                                                       ES\
1041          RETURN                                                           ES\
1042     1050 WRITE(IO3,1052) ICC
1043      710 DO 711 LL = 1, N                                                 ES\
1044          I2 = IROW (LL,1)                                                 ES\
1045      711 IROW(I2,2) = LL                                                  ES\
1046          IF(RCOTI) 607, 652, 607                                          ES\
1047     1052 FORMAT(///23H ****** WARNING ******    ,'  SUBROUTINE EIGVEC HAS ES\
            1FOUND AN EIGENVALUE OF APPARENT MULTIPLICITY',                   ES\
            1                                          I4,/23X,'  COMPUTATION OF EIES\
            2GENVECTOR(S) CONTINUES AT USER S OPTION'//)                      ES\
1048      101 FORMAT(38HOMORE THAN 15 LOOPS FOR EIGENVECTOR OF,2E12.4,        ES\
            2   14H DIFFERENCE OF,E12.4)                                      ES\
1049      102 FORMAT(16HO****WARNING****  , I4, 71H ZEROS ON DIAGONAL OF FACTOREDES\
            1 MATRIX.  CHECK FOR MULTIPLE EIGENVALUES./20X,                   ES\
            2' SUBROUTINE EIGVEC WILL NOT PERFORM COMPUTATION FOR THIS EIGENVECES\
            3TOR '//)                                                         ES\
1050          END
        /DATA
```